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椭圆曲线$y^2 = x^3 + 2\,437x–4\,882$的整数点*

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摘要:研究椭圆曲线$y^2 = x^3 + 2\,437x–4\,882$的整数点问题。通过运用初等数论、Pell方程及四次丢番图方程的理论,系统分析该曲线的整数点分布。首先将原方程分解为$y^2 = (x–2)(x^2 + 2x + 2\,441)$进而对最大公约数$d$的不同取值进行分类讨论,结合若干关键Pell方程无解的结论,排除了所有非平凡整数解存在的可能性。最终证明该椭圆曲线仅有唯一的整数点$(x,y) = (2,0)$,完善了此类椭圆曲线整数点问题的理论结果。

关键词:椭圆曲线;整数点;初等方法;Pell方程
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Integral points on the elliptic curve $y^2=x^3+2\,437x–4\,882$*

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Abstract: This paper investigates the integer points on the elliptic curve $y^2 = x^3 + 2\,437x–4\,882$. By applying theories from elementary number theory, Pell's equations, and quartic Diophantine equations, the distribution of integer points on this curve is systematically analyzed. The original equation is first decomposed into $y^2 = (x–2)(x^2 + 2x + 2\,441)$, followed by a case analysis based on different values of the greatest common divisord. Using the fact that several key Pell's equations have no solutions, the existence of all non-trivial integer solutions is ruled out. It is ultimately proved that the elliptic curve has only one integer point $(x,y) = (2,0)$, thereby enriching the theoretical understanding of integer points on such elliptic curves.

Keywords: elliptic curve; integer points; elementary methods; Pell's equation
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0　引　言

椭圆曲线的整数点问题一直受到数论学者的广泛关注,当$m = 31$时,Zagier[1]提出了椭圆曲线

$$y^2 = x^3 + (m–4)x–2m \tag{1}$$

上的最大整数点是$(x,y)=(28\,844\,402,\pm 154\,914\,585\,540)$。2009年Zhu和Chen[2]运用理想整环及p-adic分析方法找到了曲线(1)的所有整数点。2010年吴华明[3]利用四次丢番方程的一些结果证明了曲线(1)有整数点$(x,y) = (2,0)$和$(x,y) =$

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(28 844 402, ±154 914 585 540);同年 He 和 Zhang[4] 利用四次不定方程的结论给出了曲线(1)上的所有整数点。2013 年 Karaatli 和 Keskin[5]结合广义斐波那契序列理论给出了相同的结果。

后来,一些学者对这类椭圆曲线进行了推广。Yang 和 Fu[6]运用二次和四次丢番方程,以及初等数论的相关知识证明了当 $n>1$ 且 $6n^2-1$ 和 $12n^2+1$ 为奇素数时,椭圆曲线 $y^2=x^3+(36n^2-9)x+2(36n^2-5)$ 仅有整数点 $(2,0)$。管训贵[7]运用初等数论方法对曲线(1)进行了推广,设 $m=36s^2-5$ 是素数,当 $m=31$ 时,椭圆曲线(1)仅有整数点 $(x,y)=(2,0)$ 和 $(x,y)=(28\,844\,402,\pm154\,914\,585\,540)$;当 $m\neq31$ 时,椭圆曲线(1)仅有整数点 $(x,y)=(2,0)$。Zhu 和 Chen[8-9]将丢番图方程放到理想整环中去考虑,给出了 $y^2=(x+a)(x^2-ax+b),a,b\in\mathbf{Z},a^2<4b$ 上所有的积分点。Zhao[10]利用初等方法证明了椭圆曲线 $y^2=x^3+19x+46$ 无整数点。万飞等[11]利用同余和二次剩余理论给出了椭圆曲线 $y^2=x^3+(r-36)x+6r$ 的正整数点。马江等[12]运用初等方法证明了椭圆曲线 $y^2=x^3-75x-158$ 仅有整数点 $(x,y)=(2,0)$。崔宝军[13]结合 Pell 方程理论给出了椭圆曲线 $y^2=x^3+99x-460$ 的整数点。关于椭圆曲线问题的其他研究可见文献[14-19]。

本文运用 Pell 方程及四次丢番图方程的有关结果给出了 $m=2\,441$ 时,椭圆曲线(1)的整数点。

定理 椭圆曲线

$$y^2=x^3+2437x-4\,882 \tag{2}$$

仅有整数点 $(x,y)=(2,0)$。

1 主要引理

引理 1 丢番图方程

$$u^2-31v^2=-2\,440 \tag{3}$$

无整数解。

证明 丢番图方程 $x^2-31y^2=1$ 的基解是 $(x_1,y_1)=(1\,520,273)$,设 (u_1,v_1) 为方程(3)的基解,则满足

$$0<v_1\leq\frac{273\sqrt{2\,440}}{\sqrt{2(1\,520-1)}}<245,$$

将 $v=1,2,3,\cdots,245$ 代入可得对应的 u 均不是整数,故方程(3)无整数解。

引理 2 丢番图方程

$$u^2-79v^2=-2\,440 \tag{4}$$

无整数解。

证明 丢番图方程 $x^2-79y^2=1$ 的基解是 $(x_1,y_1)=(80,9)$,设 (u_1,v_1) 为方程(4)的基解,则满足

$$0<v_1\leq\frac{9\sqrt{2\,440}}{\sqrt{2(80-1)}}<36,$$

将 $v=1,2,3,\cdots,36$ 代入可得对应的 u 均不是整数,故方程(4)无整数解。

引理 3[20] 设 $D>0$,且不是完全平方数,(V_1,U_1) 为丢番图方程 $x^2-Dy^2=1$ 的基解且满足 $V_n+U_n\sqrt{D}=(V_1+U_1\sqrt{D})^n$,则丢番图方程

$$x^2-Dy^4=1 \tag{5}$$

最多存在 2 组正整数解 (X,Y),且满足:

(1)如果方程有 2 组正整数解 (X_1,Y_1) 和 (X_2,Y_2),$Y_1<Y_2$,则 $Y_1^2=U_1,Y_2^2=U_2$,除非 $D=1\,785$ 或 $D=16\times1\,785$,在这种情况下 $Y_1^2=U_1$ 和 $Y_2^2=U_4$;

(2)如果方程只有 1 组正整数解 (X,Y),则 $Y^2=U_l$,其中 $U_1=lv^2$,(l 为无平方因子的整数,v 为整数)并且 $l=1$、$l=2$ 或 $l=p$(p 是某个满足 $p\equiv3(\mathrm{mod}4)$ 的素数)。

引理 4 丢番图方程

$$X^2-9\,796Y^4=1 \tag{6}$$

无整数解。

证明 注意到 Pell 方程 $u^2-9\,796v^2=1$ 的基本解是

$$u_1=17\,658\,084\,059\,996\,578\,885\,631\,665\,215\,199,$$
$$v_1=178\,409\,999\,371\,262\,442\,544\,156\,729\,740。$$

设 (u_n,v_n) 是 Pell 方程 $u^2-9\,796v^2=1$ 的正整数解,若原方程有正整数解,则存在正整数 n,满足

$$Y^2=v_n, \tag{7}$$

由 pell 方程解的递推关系可得

$$\begin{cases}u_{n+2}=2u_nu_{n+1}-u_n,\\u_0=1,\\u_1=17\,658\,084\,059\,996\,578\,885\,631\,665\,215\,199,\\v_{n+2}=2u_nv_{n+1}-v_n,\\v_0=0,\\v_1=178\,409\,999\,371\,262\,442\,544\,156\,729\,740。\end{cases} \tag{8}$$

由引理 3 可知,该方程至多存在 1 组解,对式(7)取模 17 得,从 $n=8$ 开始剩余序列的周期 4,即:7, 7, 0, 10,且当 $n\equiv0,1,3(\mathrm{mod}4)$ 时,$Y^2\equiv7,7,10(\mathrm{mod}17)$,矛盾,仅剩 $n\equiv2(\mathrm{mod}4)$ 由引理 3 可知此时必有 $n=2$,因此由式(7)可得

$$Y^2 = v_2 = 2u_1v_1。$$

经计算可知v_2不是完全平方数,此时无正整数解Y,故式(6)无整数解。

2　定理的证明

对式(2)变形有

$$y^2 = (x-2)(x^2 + 2x + 2\,441)。\tag{9}$$

注意到$x^2 + 2x + 2\,441 = (x+1)^2 + 2\,440 > 0$,故平凡整数点为$(x,y) = (2,0)$除外。以下考虑$x > 2$的情形。

令$d = \gcd(x-2, x^2 + 2x + 2\,441)$,则对$d = \gcd(x-2, 2\,449) = \gcd(x-2, 31 \times 79)$=1或31或79或2 449进行分类讨论。

情形1:$d = 1$,存在$a, b \in \mathbf{N}^+$,满足

$$x-2 = a^2, x^2 + 2x + 2\,441 = b^2, y = \pm ab,\tag{10}$$

其中$\gcd(a,b) = 1$。

由$x^2 + 2x + 2\,441 = b^2$得$b^2-(x+1)^2 = 2\,440$,即$(b + x + 1)(b-x-1) = 2\,440$。解之得$x = 50$或$x = 116$或$x = 302$或$x = 608$,代入式(9)得到的$y^2$均不是完全平方数,故情形1下无整数解。

情形2:$d = 31$,存在$a, b \in \mathbf{N}^+$,满足

$$x-2 = 31a^2, x^2 + 2x + 2\,441 = 31b^2, y = \pm 31ab,\tag{11}$$

其中$\gcd(a,b) = 1$。

结合$x-2 = 31a^2, x^2 + 2x + 2\,441 = 31b^2$可得

$$(31a^2 + 3)^2 + 2\,440 = 31b^2,\tag{12}$$

令$u = 31a^2 + 3, v = b$,则式(12)成为

$$u^2-31v^2 = -2\,440。\tag{13}$$

由引理1可知,此方程无解,故情形2下无整数解。

情形3:$d = 79$,存在$a, b \in \mathbf{N}^+$,满足

$$x-2 = 79a^2, x^2 + 2x + 2\,441 = 79b^2, y = \pm 79ab,\tag{14}$$

其中$\gcd(a,b) = 1$。

结合$x-2 = 79a^2, x^2 + 2x + 2\,441 = 79b^2$可得,

$$(79a^2 + 3)^2 + 2\,440 = 79b^2,\tag{15}$$

令$u = 79a^2 + 3, v = b$,则式(15)为,

$$u^2-79v^2 = -2\,440。\tag{16}$$

由引理2可知,此方程无解,故情形3下无整数解。

情形4:$d = 2\,449$,存在$a, b \in \mathbf{N}^+$,满足

$$x-2 = 2\,449a^2, x^2 + 2x + 2\,441 = 2\,449b^2, \quad y = \pm 2\,449ab;\tag{17}$$

其中$\gcd(a,b) = 1$

结合$x-2 = 2\,449a^2, x^2 + 2x + 2\,441 = 2\,449b^2$有

$$(3a^2 + 1)^2 + 2\,440a^4 = b^2。\tag{18}$$

若$2 \nmid a$,则$(3a^2 + 1)^2 \equiv 0 \pmod{16}$,$a^4 \equiv 1 \pmod{16}$,而$2\,440 \equiv 0 \pmod 8$,$2\,440 \equiv 8 \pmod{16}$得$8 \mid b^2$,从而$4 \mid b$。对式(18)模16,有$8 \equiv 0 \pmod{16}$,矛盾。故$2 \mid a$。令$a = 2c, c \in \mathbf{N}^+$,代入式(18)得

$$(12c^2 + 1)^2 + 39\,040c^4 = b^2,\tag{19}$$

变形得到

$$(b + 12c^2 + 1)(b-12c^2-1) = 39\,040c^4。\tag{20}$$

设$l = \gcd(b + 12c^2 + 1, b-12c^2-1)$,由式(20)得,$12c^2 + 1$与$b$同为奇数,故$l$为偶数。若$l > 2$,注意到$l \mid 2b$且$l \mid 2(12c^2 + 1)$,故$\frac{l}{2}$为奇数,设$p$是$\frac{l}{2}$的奇素因子,由式(20)知,$p \mid b$,$p \mid (12c^2 + 1)$,$p^2 \mid 39\,040c^4$,又因为$37\,760 = 2^7 \times 5 \times 61$,故$p \mid c$。但这与$p \mid (12c^2 + 1)$矛盾。故$l = 2$。

存在$f, g, t \in \mathbf{N}^+$满足

$$\begin{cases} b + (12c^2 + 1) = \dfrac{2^7 \times 5 \times 61}{t} f^4, \\ b-(12c^2 + 1) = tg^4, \\ c = fg, \\ \gcd(f,g) = 1, \end{cases}\tag{21}$$

$t = 2, 10, 64, 122, 320, 610, 3\,904, 19\,520$,下面对这8种情形进行分类讨论:

$t = 2$,给式(21)前2个式子做差有,有

$$12c^2 + 1 = 9\,760f^4-g^4,\tag{22}$$

故$2 \nmid g$,此时$1 \equiv 3 \pmod 4$矛盾,故$t = 2$时无整数解。

$t = 10$,给式(21)前2个式子做差有,有

$$12c^2 + 1 = 1\,952f^4-5g^4,\tag{23}$$

故$2 \nmid g$,但此时$1 \equiv 3 \pmod 4$矛盾,故$t = 10$时无整数解。

$t = 64$,给式(21)前2个式子做差有,有

$$12c^2 + 1 = 305f^4-32g^4,\tag{24}$$

将$c = fg$代入得

$$(305f^2-6g^2)^2-9\,796g^4 = 305,\tag{25}$$

令$X = |305f^2-6g^2|$,$Y = g$,则式(25)成为

$$X^2-9\,796Y^4 = 305,\tag{26}$$

对式(26)取模31,得$X^2 \equiv 26 \pmod{31}$,注意到$\left(\frac{26}{31}\right) = -1$,显然矛盾。

$t = 122$,对式(21)前2个式子做差有,

$$12c^2 + 1 = 160f^4-61g^4,\tag{27}$$

显然$2 \nmid g$,取模16有$1, 13 \equiv 3 \pmod{16}$,显然矛盾。

$t = 320$,给式(21)前2个式子做差有,有

$$12c^2 + 1 = 61f^4-160g^4,\tag{28}$$

代入$c=fg$得

$$(61f^2-6g^2)^2-9\,796g^4=61, \tag{29}$$

令$X=|61f^2-6g^2|$,$Y=g$,则式(29)成为

$$(X)^2-9\,796Y^4=61, \tag{30}$$

取模31,有$X^2\equiv 30(\mathrm{mod}31)$,注意到$\left(\frac{30}{31}\right)=-1$,这显然矛盾。

$t=610$,给式(21)前2个式子做差有,有

$$12c^2+1=32f^4-305g^4, \tag{31}$$

显然$2\nmid g$,取模16得$1,13\equiv -1(\mathrm{mod}16)$,矛盾。

$t=3\,904$,给式(21)前2个式子做差有,有

$$12c^2+1=5f^4-1\,952g^4, \tag{32}$$

显然$2\nmid f$。取模16得到$1,13\equiv 5(\mathrm{mod}16)$,这显然矛盾。

$t=19\,520$,给式(21)前2个式子做差有,有

$$12c^2+1=f^4-9\,760g^4, \tag{33}$$

代入$c=fg$得

$$(f^2-6g^2)^2-9\,796g^4=1, \tag{34}$$

令$X=|f^2-6g^2|$,$Y=g$,则式(34)成为

$$X^2-9\,796Y^4=1, \tag{35}$$

由引理4可知情形4下无整数解。综上,式(2)仅有整数点$(x,y)=(2,0)$。

3 结束语

椭圆曲线的整数点问题一直是数论研究的热点问题,本文运用Pell方程及四次丢番图方程的有关结果给出椭圆曲线$y^2=x^3+2\,437x-4\,882$仅有整数点$(x,y)=(2,0)$,可为进一步探究该类椭圆曲线的整数点问题提供借鉴。

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