DOI：10.19789/j.1004-9398.2026.01.001　　**Open Access**
文献引用：戎小春.正截面曲率流形的基本群[J].首都师范大学学报（自然科学版），2026，47（1）：1-13+27. RONG X C. Fundamental groups of manifolds of positive sectional curvature[J]. Journal of Capital Normal University（Natural Science Edition），2026，47（1）：1-13+27.

正截面曲率流形的基本群*

戎　小　春**

（首都师范大学数学科学学院，北京　100048；罗格斯大学新布朗斯威克分校数学科学学院，新泽西州　08903）

摘要：本文是关于正截面曲率紧致流形基本群的综述报告。回顾有关经典的成果，介绍关于几乎循环基本群猜想的提出背景，即一个 n-维正截面曲率紧致流形的基本群，含有一个指标不超过1个只依赖 n 的常数的循环子群。介绍关于此猜想的历史进展，包括一些最新的成果。

关键词：正界面曲率；几乎循环基本群猜想；等变Gromov-Hausdorff收敛；亚历山大空间几何
中图分类号：O186.1　　**文献标志码**：A

Fundamental groups of manifolds of positive sectional curvature*

RONG Xiaochun**

（School of Mathematical Sciences，Capital Normal University，Beijing　100048；Mathematics Department，Rutgers University New Brunswick，State of New Jersey　08903）

Abstract：Let M be a compact n-manifold of positive sectional curvature. We will review classical results on the fundamental group of M，a motivation on the $c(n)$-cyclic conjecture that the fundamental group of M contains a cyclic subgroup of index bounded above by $c(n)$，a constant depending only on n，and we will survey partial results（up to date）on the $c(n)$-cyclic conjecture.

Keywords：positive sectional curvature；conjecture on almost cyclic fundamental groups；equivariant Gromov-Hausdorff convergence；Alexandrov geometry
CLC：O186.1　　**DC**：A

0　Overview

Let M be a complete n-manifold of nonnegative Ricci curvature. The generalized Margulis lemma says that the fundamental group of M is $c(n)$-nilpotent i.e.，$\pi_1(M)$ contains a nilpotent subgroup of index $\leqslant c(n)$，a constant depending only on n，and if M is compact，then $\pi_1(M)$ has generators of number$\leqslant a(n)$[1-3]，which fails to hold if M is not compact[4-5]. Moreover，a counterexample to a well-known conjecture was found in reference [6]；the conjecture says that if M is compact，then $\pi_1(M)$ should be $c(n)$-abelian.

Let M be a compact manifold of positive Ricci curvature，normalized to $\mathrm{Ric}_M \geqslant (n-1)$. Then the diameter，$\mathrm{diam}(M) \leqslant \pi$[7]，thus $\pi_1(M)$ is finite. On the other hand，any finite group Γ can be homomorphically embedded into $\mathrm{Spin}(2|\Gamma|)$，while any

收稿日期：2024-08-04
*国家自然科学基金项目（11821101）；北京市自然科学基金项目（Z190003）；首都师范大学基金项目
**通信作者：rong@math.rutgers.edu

bi-invariant metric on Spin$(2|\Gamma|)$ has positive Ricci curvature and nonnegative sectional curvature.

For a complete manifold M of positive sectional curvature, $\sec_M > 0$, there is a much strong restriction on $\pi_1(M)$ than a $c(n)$-nilpotence. If M is not compact, then M is diffeomorphic to a Euclidean space[8-9], and if M is compact, then we have following strong constraint on the structure of $\pi_1(M)$.

Theorem 0.1[10] Let M be a complete n-manifold of $\sec_M \geqslant 1$.

(0.1.1) If $n = 2m$, then M is simply connected, or not orientable.

(0.1.2) If $n = 2m + 1$, then M is orientable.

Note that theorem 0.1 gives an obstruction for a compact non-simply connected manifold M of $\sec_M \geqslant 0$ which does not support a metric of positive sectional curvature.

Corollary 0.2 Let $\mathbb{R}P^m$ denote the real projective m-space of constant sectional curvature one. Then the metric product, $M = \mathbb{R}P^2 \times \mathbb{R}P^{n-2}$ $(n \geqslant 4)$, admits no metric of positive sectional curvature.

In the study of the topology of a compact manifold of positive sectional curvature, a somewhat frustrating reality is that no single topological obstruction has been known for a compact simply connected manifold of nonnegative sectional curvature which does not support a metric of positive sectional curvature; e.g., whether the metric product to two unit spaces admits a metric of positive sectional curvature (the Hopf's conjecture).

By theorem 0.1, if $\dim(M) = 2m$, then $\pi_1(M) = \{e\}$ or $\mathbb{Z}_2$ i.e., no finite group except $\mathbb{Z}_2$ can be realized in each even-dimension. On the other hand, in each odd-dimension, the fundamental groups of spherical space form contains infinitely many isomorphic classes[11].

It is natural to expect that for $\dim(M) = 2m + 1$, more can be said than that M is orientable. In reference [12], Chern asked for M of $\sec_M \geqslant 1$, whether any abelian subgroup of $\pi_1(M)$ is cyclic; a partial motivation seems to be that by Preissmann theorem, if M is a compact manifold of negative sectional curvature, then any abelian subgroup of $\pi_1(M)$ is cyclic. Note that Chern's conjecture is true for spherical space forms.

The following results provided negative answers to the Chern's conjecture in dimensions 7 and 13, the only dimensions in which infinitely many compact simply connected manifolds of positive sectional curvature have been found.

Theorem 0.3[13-14] (0.3.1) For each integer $h > 0$, there is a 7-manifold M_h of $\sec_{M_h} > 0$ such that $\pi_1(M_h) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2h}$, or $\pi_1(M_h) \cong \mathbb{Z}_3 \times \mathbb{Z}_{3h}$.

(0.3.2)[15] For each integer $h > 0$, there is a 13-manifold M_h of $\sec_{M_h} > 0$ such that $\pi_1(M_h) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2h}$, or $\pi_1(M_h) \cong \mathbb{Z}_3 \times \mathbb{Z}_{3h}$.

Prior to theorem 0.3, partial positive results on Chern's conjecture had been obtained, which suggested the following.

Conjecture 0.4 ($c(n)$-cyclic[16]**)** Let M be a compact n-manifold of positive sectional curvature. Then $\pi_1(M)$ is $c(n)$-cyclic i.e., $\pi_1(M)$ has a cyclic subgroup of index $\leqslant c(n)$, a constant depending only on n.

In conjecture 0.4, a bound on index depending on n is of necessary (e.g., fundamental groups of spherical n-forms, see reference [11]). Conjecture 0.4 proposes a dimension-related structural constraint on a fundamental group; e.g., conjecture 0.4 implies that for any integer $h > c(9)$, the metric product of spherical space forms, $M = S^3_1 \times (S^3_1/\mathbb{Z}_h) \times (S^3_1/\mathbb{Z}_h)$ (orientable), admits no metric of positive sectional curvature.

The next two results are partial verification of conjecture 0.4.

Theorem 0.5[17-18] Let M be a compact n-manifold of $\sec_M > 0$. If the Riemannian universal covering admits an isometric S^1-action, then $\pi_1(M)$ is $c(n)$-cyclic.

Note that theorem 0.5 was obtained in reference [17] by assuming that $\tilde{M}$ admits an isometric torus T^k-action $(k \geqslant 1)$ whose orbits are preserved by the $\pi_1(M)$-action, and the $\pi_1(M)$-invariance is always satisfied for a subgroup of $\pi_1(M)$ of index bounded

above by a constant depending only on n(see lemma 1.1 in reference [19]).

The following is a corollary of theorem 0.5.

Corollary 0.6 The fundamental group of any spherical n-space form is $c(n)$-cyclic.

Roughly speaking, theorem 0.5 says that in the presence of an isometric S^1-action, one is able to convert the strong constraint on $\pi_1(M)$ in even dimension to partially verify conjecture 0.4 (in odd dimension).

Indeed, a long standing problem asks if a compact simply connected manifold M which admits a metric of positive sectional curvature also supports a metric of positive sectional curvature with a S^1-symmetry.

On the other hand, in the following situations there are (weak) positive answers, using which we obtain partial verification of conjecture 0.4.

Theorem 0.7 Let M be a complete n-manifold of $\sec_M \geqslant 1$.

(0.7.1)[16] If M satisfies that $\sec_M \leqslant K$, then $\pi_1(M)$ is $c(n, K)$-cyclic.

(0.7.2)[20] If M satisfies that $\mathrm{vol}(\tilde{M}) \geqslant v > 0$, then $\pi_1(M)$ is $c(n, v)$-cyclic.

We point it out that a proof of (0.7.1) replies on finding on a sufficiently collapsed M a nearby metric of positive sectional curvature with a S^1-symmetry, to which applying theorem 0.5. In the proof of (0.7.2), we fail to get a (nearby) metric of positive sectional curvature and a S^1-symmetry, rather by a standard compactness argument we obtain an Alexandrov space of positive curvature and an isometric T^k-symmetry ($k \geqslant 1$), so tools from Alexandrov geometry[21-23] are required.

1 Proof of theorem 0.5

A proof of theorem 0.5 needs some preparation.

Lemma 1.1[19] Let M be a compact n-manifold of finite $\pi_1(M)$. If $\tilde{M}$ admits an isometric S^1-action, then $\pi_1(M)$ contains a subgroup Γ such that the Γ-action commutes with the S^1-action and $[\pi_1(M):\Gamma] \leqslant b(n)$.

Proof Note that $S^1, \Gamma < \mathrm{Isom}(\tilde{M})$ which is a compact Lie group, and S^1 is contained in a maximal torus T^k. By the classification of compact Lie groups, a finite covering group of the identify component, $\mathrm{Isom}(\tilde{M})_0$, of number depending on n, is isomorphic to $T^k \times G_s$, where G_s is a product of semi-simple Lie groups. Without loss of generality, we may assume $\mathrm{Isom}(\tilde{M})_0$ is simply connected. Consider the induced map, $\phi: \Gamma \to \mathrm{Aut}(T^k)$ by conjugation, where T^k is a maximal torus. Again, by the classification of semi-simple Lie group, because $\dim(\mathrm{Isom}(\tilde{M})) \leqslant \frac{n(n+1)}{2}$, $|N(T^k)/T^k| \leqslant s(n)$, thus $|\mathrm{Im}(\phi)| \leqslant s(n)!$. Then $\Gamma = \ker\phi$ has the desired property.

Lemma 1.2 (isotropy groups[24]**)** Assume that a compact Lie group G acts isometrically on a complete manifold M. If $\gamma(t)$ is a minimal geodesic from $G(\gamma(0))$ to $G(\gamma(1))$, If $\gamma(t)$ is a minimal geodesic such that $d\big(\gamma(0), \gamma(1)\big) = d\big(G(0), G(s)\big)$, then for all $0 < t < 1$, $G_{\gamma(t)}$ is a subgroup of both $G_{\gamma(0)}$ and $G_{\gamma(1)}$. In particular, $G_{\gamma(t)}$ is independent of $0 < t < 1$.

Proof Let $x_0 = \gamma(0), x_1 = \gamma(1)$, and $x = \gamma(t)$, $0 < t < 1$. We need to show that $G_x(x_0) = x_0$ and $G_x(x_1) = x_1$. We argue by contradiction. Assume $\alpha \in G_x$ such that $\alpha(x_0) \neq x_0$. Then there is a geodesic triangle whose two sides are $\alpha(\gamma|_{[0,t]})$ and $\alpha(\gamma|_{[t,1]})$. Because there is a corner at the vertex x (otherwise, we see a bifurcation of geodesics at $\gamma(t)$), $d(\alpha(x_0), x_1) < d(\alpha(x_0), x) + d(x, x_1) = d(x_0, x) + d(x, x_1) = d(x_0, x_1) = d(G(x), G(y))$, a contradiction.

Theorem 1.3 (equivariant Sygne version[18]**)** Let M be a compact simply connected manifold of positive sectional curvature. Assume that M admits an isometric S^1-action. If ϕ is an isometry of M that commutes with the S^1-action, then ϕ preserves some circle S^1-orbit.

Proof Case 1. Assume that ϕ has a fixed point component $p \in F_0 \subset F(M, \phi)$. Because $S^1(F_0) = F_0$, without loss of generality one may assume that $S^1|_{F_0}$ is trivial. Consider the isotropy representation of $\mathrm{d}\phi$ and $\mathrm{d}S^1$ on the unit sphere, $S_1^{2n-2}(0) \subset T_pM$. By applying

induction on $(S_1^{2n-2}(0), \mathrm{d}S^1, \mathrm{d}\phi)$, one concludes that $\mathrm{d}\phi$ preserves a circle $\mathrm{d}S^1$-orbit. Because $\exp_p$ is a local diffeomorphism and commutes with the differential actions, the desired result follows.

Case 2. Assume that ϕ has no fixed point. Because ϕ commutes with the S^1-action, ϕ descends to an isometry ϕ^* on the orbit space $M^* = M/S^1$ (which may not be effective). Because ϕ has no fixed point, ϕ preserves a circle S^1-orbit iff ϕ^* has a fixed point in M^*.

We proceed by induction on $\dim(M) = 2n+1$. For $n=1$, M^* is an orientable 2-surface of a finite fundamental group, thus M^* is homeomorphic to either a two sphere or a two-disk. Because ϕ^* preserves the orientation of M^*, ϕ^* has a fixed point on M^*.

Assume that if $\dim(M) \leqslant 2n-1$, ϕ^* has a fixed point.

For $\dim(M) = 2n+1$, if ϕ^* has no fixed point, then let $p^* \in M^*$, $0 < d^*(p^*, \phi^*(p^*)) \leqslant d^*(x^*, \phi^*(x^*))$ for all $x^* \in M^*$. Fixing $p \in \mathrm{proj}^{-1}(p^*)$, let σ denote a geodesic from p to $S^1(\phi(p))$ such that $\mathrm{length}(\sigma) = d^*(p^*, \phi^*(p^*))$. By lemma 1.2, for $0 < t < 1$, the isotropy group $S^1_{\sigma(t)} = H$ is independent of t, which is a subgroup of $S^1_p = S^1_{\phi(p)}$.

a. Assume that $\{e\} \neq H$ is finite. Let $F_0 \subseteq F(M, H)$ be a H-fixed point component. Because $x, \phi(x) \in F_0$, $\phi(F_0) = F_0$. We then apply induction on n, starting with $n=2$, thus $\dim(F_0) = 3$. By the inductive assumption on $(F_0, S^1|_{F_0}, \phi)$, we conclude that ϕ^* has a fixed point in F_0/S^1, a contradiction.

b. Assume that $H = \{e\}$. Because $S^1(p)$ and $S^1(\phi(p))$ are principal S^1-orbits, all $S^1(\alpha(t))$ are principal S^1-orbits. Because the union of all principal S^1-orbits M_0 is open, M_0^* is an even-dimensional open manifold of positive sectional curvature. Because the S^1-action preserves the orientation on M_0, M_0^* is orientable, and because ϕ is an orientation-preserving isometry, ϕ^* is an orientation-preserving isometry on M_0^*. Following the standard Synge-type argument as in the proof of theorem 0.1, the displacement function of ϕ^* cannot achieve a (local) minimum at p^*, a contradiction.

c. Assume that $H = S^1$. Let F_0 denote the H-fixed point component at p; which is a closed totally geodesic $(2k+1)$-submanifold. Then $\sigma(t) \subset F_0$, thus $\phi(F_0) = F_0$. Let $T_p^\perp(\sigma) \subset T_pM$ denote the orthogonal complement of $\sigma'(0)$, and let $\psi = \mathrm{d}\phi^{-1} \circ P_\sigma : T_p^\perp(\sigma) \to T_p^\perp(\sigma)$, where P_σ is the parallel translation along σ from $\sigma(0)$ to $\sigma(1)$. Then ψ is a linear isometry on $T_p^\perp(\sigma)$, and we claim that there is $V \in T_p^\perp(\sigma)$ such that $\psi(V) = \mathrm{d}t(V)$ for some $t \in S^1$.

Assuming the claim, we will derive a contradiction. Let $V(t)$ denote the parallel vector field along σ with $V(0) = V$ and let $\sigma_\epsilon(t) = \exp_{\sigma(t)} \epsilon V(t)$ for some fixed small ϵ. By the second variation formula of arc length, we have that $\mathrm{length}(\sigma_\epsilon) < \mathrm{length}(\sigma)$. Put $x = \exp_p \epsilon V$, then

$$\phi^*(x^*) = (\phi(tx))^* = (\phi(t\exp_p \epsilon V))^* =$$
$$(t\phi(\exp_p \epsilon V))^* = (t\exp_{\phi(p)} \epsilon \mathrm{d}\phi(V))^* =$$
$$(\exp_{t\phi(p)} \epsilon \mathrm{d}t \circ \mathrm{d}\phi(V))^* = (\exp_{\sigma(1)} \epsilon V(1))^* = \sigma_\epsilon(1)^*,$$

and therefore

$$\mathrm{d}^*(x^*, \phi^*(x^*)) \leqslant \mathrm{length}(\sigma_\epsilon^*) \leqslant \mathrm{length}(\sigma_\epsilon) <$$
$$\mathrm{length}(\sigma) = \mathrm{d}^*(p^*, \phi^*(p^*)),$$

a contradiction.

Because $\sigma(t) \subset F_0$, the S^1-action commutes with the parallel translation P_σ (from $\mathrm{d}t(P_{\sigma(t)}v)$ being a parallel field, one sees that $\mathrm{d}t(P_{\sigma(t)}v) = P_{\sigma(t)}\mathrm{d}t(v)$), and therefore ψ commutes with the S^1-action on $T_p^\perp(\sigma)$. Let S^{2n-2} denote the unit ball in $T_p^\perp(\sigma)$. Then ψ induces an isometric action on S^{2n-2} commuting with the S^1-action. Since both $\mathrm{d}\phi$ and the S^1-action on S^{2n-2} are orientation-preserving, by the inductive assumption, ψ^* has a fixed point in S^{2n-2}/S^1 and thus there is a $V \in S^{2n-2}$ such that $\psi(V) = \mathrm{d}t(V)$ for some $t \in S^1$.

By scaling, we may assume that $\sec_M \geqslant 1$. Then orbit space, M/S^1, is a metric space satisfying Toponogov triangle comparison with respect to S_1^2; M/S^1 is called an Alexandrov space of curvature $\geqslant 1$ (see section 2). In reference [23], theorem 0.1 has been extended to M/S^1 (see theorem 2.11), and theorem 1.3 follows easily from theorem 2.11 (see section 2).

Our proof of theorem 0.5 also uses the following two fundamental results.

Theorem 1.4 (fixed points[25]) Let M be a compact manifold, and let $G = T^k$ or $\mathbb{Z}_p^k$ (p a prime) act effectively on M, then

$$\sum_q \operatorname{rank}\,(H_q(F(M;\ G);\ \ell)) \leqslant \sum_q \operatorname{rank}\,(H_q(M;\ \ell)),$$

where ℓ is a field for $G = T^k$, and $\ell = \mathbb{Z}_p$ for $G = \mathbb{Z}_p^k$.

Theorem 1.5 (Betti number estimate[26]) Let M be a compact n-manifold of $\sec_M \geqslant 0$. Then for any field ℓ, the total Betti number of M,

$$b(M) = \sum_{k=0}^{n} \operatorname{rank}\,(H_k(M;\ \ell)) \leqslant b(n).$$

Theorem 1.6 (Margulis lemma[27]) Given n, there exist constants $\epsilon(n), c(n) > 0$ such that if a compact n-manifold M satisfies that $\sec M > -1$ and $\operatorname{diam}(M) < \epsilon(n)$, then $\pi_1(M)$ is $c(n)$-nilpotent i.e., contains a nilpotent subgroup of index $\leqslant c(n)$.

Proof of theorem 0.5 By lemma 1.1, without loss of generality, we may assume that M admits an isometric S^1-action. Consider the lifting $\tilde{S}^1$-action on $\tilde{M}$, which commutes with the $\pi_1(M)$-action.

Case 1. Assume the $\tilde{S}^1$-action has a finite isotropy group at $\tilde{x}_0 \in \tilde{M}$. Without loss of generality, we assume $\mathbb{Z}_p < \tilde{S}^1$ (p a prime), $\mathbb{Z}_p(\tilde{x}_0) = \tilde{x}_0$. Let $F(\tilde{M}, \mathbb{Z}_p) = \bigcup_{j=0}^{k} F_j$ denote the $\mathbb{Z}_p$-fixed point set, F_0 denotes the component at $\tilde{x}_0$. Then each F_j is a closed totally geodesic submanifold of even-codimension, thus each F_j is an odd-dimension compact manifold of positive sectional curvature on which $\tilde{S}^1$ acts.

Let $\Lambda < \pi_1(M)$ denote the subgroup preserving F_0. We now proceed by induction on n, starting with $n = 3$. One may apply the classification of a Seifert 3-manifolds[26], or assume that M is a 3-spherical[11], or simply runs the following proof in dimension 3 (which is much simple than in higher dimensions).

By induction on n we may assume that $\pi_1(F_0/\Lambda)$ contains a cyclic subgroup of index $\leqslant a(n)$, thus Λ has a normal cyclic subgroup $\mathbb{Z}_h$ of index $\leqslant a(n)$. Then $[\Lambda : \Lambda \cap \mathbb{Z}_h] \leqslant [\pi_1(F/\Lambda) : \mathbb{Z}_h] \leqslant a(n)$.

It remains to show that $[\pi_1(M) : \Lambda] \leqslant b(n)$. For $\alpha, \beta \in \pi_1(M)$, $\alpha\Lambda = \beta\Lambda$ iff $\beta^{-1}\alpha \in \Lambda$ iff $\beta^{-1}\alpha(F_0) = F_0$ iff $\alpha F_0 = \beta F_0$. Because Γ commutes with $\mathbb{Z}_p$, αF_0 is a component of $F(\tilde{M}, \mathbb{Z}_p)$, thus from theorem 1.4 and 1.5,

$$[\pi_1(M) : \Lambda] \leqslant \#\{\text{components of } F(\tilde{M}, \mathbb{Z}_p)\} \leqslant \operatorname{rank}(H_*(F(\tilde{M}, \mathbb{Z}_p);\ \mathbb{Z}_p)) \leqslant \operatorname{rank}(H_*(\tilde{M};\ \mathbb{Z}_p)) \leqslant b(n).$$

Case 2. Assume that the $\tilde{S}^1$-on $\tilde{M}$ is semi-free i.e., $\tilde{S}^1$ is the only possible isotropy subgroup. Without loss of generality, we may assume that $\pi_1(M)$ is nilpotent (theorem 1.6).

Let $H = c(\pi_1(M)) \cap = \langle \alpha \rangle$, where $c(\pi_1(M))$ denotes the center of $\pi_1(M)$. We may assume $\gamma \in \pi_1(M) - H$, and by theorem 1.3, γ preserves some $\tilde{S}^1$-orbit, $\tilde{S}^1(\tilde{x}_0)$. Because $\tilde{S}^1$ -action is semi-free, there is $t_0 \in \tilde{S}^1$ such that $t_0\gamma$ fixes all points in $\tilde{S}^1(\tilde{x}_0)$.

We first observe a simple case that $\pi_1(M)$ is abelian. In this case, we may assume that γ is chosen with a prime order $p > 1$, and treating $\langle t_0\gamma \rangle$ as $\mathbb{Z}_p$ in case 1, the proof goes through, and $\pi_1(M)$ is $c(n)$-cyclic.

In general, $\pi_1(M)$ is nilpotent and $H \neq \pi_1(M)$. Consider $(\hat{M}, \hat{S}^1, \hat{\pi}_1(M))$, where $\hat{S}^1 = \tilde{S}^1/H$ acts freely and isometrically on $\hat{M} = \tilde{M}/H$, commuting with the free and isometric $\hat{\pi}_1(M) = \pi_1(M)/H$-action on $\hat{M}$.

Because $\hat{\pi}_1(M)$ is nilpotent, we may assume $\hat{\beta} \in c(\hat{\pi}_1(M))$ with order a prime number $p > 1$. Note that $\hat{\beta}$ is not contained in $\hat{S}^1$; otherwise a representative β in $\hat{\pi}_1(M)$ is contained in $\tilde{S}^1$, a contradiction.

Because $\hat{\beta}$ commutes with the $\hat{S}^1$-action, $\hat{\beta}$ preserves a $\hat{S}^1$-orbit at $\hat{x}_0 \in \hat{M}$, thus there is a $\hat{t}_0 \in \hat{S}^1$, such that $\hat{t}_0\hat{\beta}$ fixes all points in $\hat{S}^1(\hat{x}_0)$. Because $\hat{t}_0\hat{\beta}$ commutes with $\hat{\pi}_1(M)$, one is able to apply the proof of case 1 to conclude that $\hat{\pi}_1(M)$ has a cyclic subgroup $\langle \hat{\gamma} \rangle$ of index $\leqslant c(n)$.

We may assume that the abelian subgroup $\langle \gamma | \alpha \rangle$

of $\pi_1(M)$ contains a cyclic subgroup $\mathbb{Z}_h$ of index$\leqslant c(n)$. Finally

$$\left[\pi_1(M):\mathbb{Z}_h\right]\leqslant\left[\pi_1(M):\langle\alpha,\gamma\rangle\right]\cdot\left[\langle\alpha,\gamma\rangle::\mathbb{Z}_h\right]=\left[\hat{\pi}_1(M):\langle\hat{\gamma}\rangle\right]\cdot c(n)\leqslant c(n)^2.$$

2 Preparations of proof of theorem 0.7

In this section, we will review notions and results that are required in the proof of theorem 0.7.

2.1 Equivariant Gromov-Hausdorff convergence

The notions and properties on an equivariant Gromov-Hausdorff convergence for compact metric spaces also extend to complete locally compact metric spaces. Because this paper involves compact metric spaces (except one in the proof of lemma 2.3), for simplicity we will confine ourself to compact metric spaces. The references for this subsection can be found in reference [28-29].

Given two compact metric spaces, (X,d_X) and (Y,d_Y), and $\epsilon>0$, a map, $f:X\to Y$, is called an ϵ-Gromov-Hausdorff approximation (briefly, GHA), if f is an ϵ-isometry i.e.$|\mathrm{d}_X(x_1,x_2)-\mathrm{d}_Y(f(x_1),f(x_2))|<\epsilon$ and f is ϵ-onto i. e., $f(X)\subseteq Y$ is ϵ-dense. The Gromov-Hausdroff distance of (X,d_X) and (Y,d_Y) is defined by

$$d_{\mathrm{GH}}(X,Y)=\inf\{\epsilon,\ \text{there is an}\ \epsilon-\mathrm{GHA}:f:X\to Y\}.$$

It is easy to check that $d_{\mathrm{GH}}(X,Y)=0$ iff X is isometric to Y. Let $(M_{\mathrm{et}},d_{\mathrm{GH}})$ denote the collection of isometric classes of compact metric spaces, equipped with d_{GH}.

A sequence of compact metric spaces X_i d_{GH} converges to a compact metric space X, denoted by $X_i\overset{\mathrm{GH}}{\to}X$, iff $d_{\mathrm{GH}}(X_i,X)\to 0$. For $\epsilon>0$, an ϵ-net in X is a subset, $X(\epsilon)=\{x_i\}\subset X$, such that $d_X(x_i,x_j)\geqslant\epsilon$ for all $x_i\neq x_j$, open ϵ-balls, $\{B_\epsilon(x_i),x_i\in X\}$ form an open cover for X.

Assume that $X_i\to X$. Then there is $d>0$ such that $\mathrm{diam}(X_i)\leqslant d$, and for any $\epsilon>0$, $|X_i(\epsilon)|\leqslant C(\epsilon)$, a constant depending only on ϵ. Conversely, any sequence X_i contains a convergent subsequence if X_i satisfies the following conditions:

(c_1) (uniformly bounded) $\mathrm{diam}(X_i)\leqslant d$,

(c_2) (equi-continuous) For any $\epsilon>0$, $|X_i(\epsilon)|\leqslant C(\epsilon)$.

The above implies that $(M_{\mathrm{et}},d_{\mathrm{GH}})$ is a complete metric space.

One easily extends Gromov-Hausdorff-distance to complete locally noncompact metric spaces. Two pointed metric spaces, (X,p) and (Y,q), are isometric, if there is an isometry, $f:(X,p)\to(Y,q)$, $f(p)=q$. A map, $f:(X,p)\to(Y,q)$, $f(p)=q$, is called a pointed ϵ-GHA, if $f:B_{\epsilon^{-1}}(p)\to B_{\epsilon^{-1}+\epsilon}(q)$ is an ϵ-isometry and $f(B_{\epsilon^{-1}}(p))\subset B_{\epsilon^{-1}+\epsilon}(q)$ is ϵ-dense. The pointed GH-distance between (X,p) and (Y,q) is defined by

$$d_{\mathrm{GH}}((X,p),(Y,q))=\inf\{\epsilon,\exists\ \text{pointed}\ \epsilon-\mathrm{GHA}:f:(X,p)\to(Y,q)\}.$$

Similarly to the compact case, it is easy to check that the collection of isometric classes of pointed complete locally compact metric spaces is a complete metric space; if X_i is also a length metric space, then $(X_i,p_i)\overset{\mathrm{GH}}{\to}(X,p)$ iff for every $r>0$, closed r-balls, $\overline{B}_r(p_i)\overset{\mathrm{GH}}{\to}\overline{B}_r(p)$.

By the Bishop-Gromov's relative volume comparison, one immediately verifies (c_2) for compact n-manifolds of Ricci curvature bounded blow.

Theorem 2.1 (precompactness[26]) Any sequence of complete n-manifolds $M_i(p_i\in M_i)$ of Ricci curvature $\mathrm{Ric}_{M_i}\geqslant-(n-1)$ contains a subsequence, $(M_{i_j},p_{i_j})\overset{\mathrm{GH}}{\to}(X,p)$.

Let $\mathrm{Isom}(X)$ denote the isometry group of X. There is a natural invariant metric on $\mathrm{Isom}(X)$ defined as follows: for $\alpha,\beta\in\mathrm{Isom}(X)$,

$$d_{\mathrm{Isom}(X)}(\alpha,\beta)=\max\{d_X(\alpha(x),\beta(x)),x\in X\}.$$

Because (X,d_X) is a compact metric space, $(\mathrm{Isom}(X),d_{\mathrm{Isom}(X)})$ is a compact metric space, and if $\{X_i\}$ satisfies (c_1) and (c_2), then $\{\mathrm{Isom}(X_i)\}$ also satisfy (c_1) and (c_2) (may have a different bound $C(\epsilon)$). Consequently, passing to a subsequence, $\mathrm{Ismo}(X_i)\overset{\mathrm{GH}}{\to}G$, which is a closed subgroup of $\mathrm{Isom}(X)$.

Given (X,Γ) and (Y,G), X and Y are compact

metric spaces, Γ and G are closed group of isometries on X and Y, respectively, a pair of maps, (f, ϕ): $(X, \Gamma) \to (Y, G)$, is called an ϵ-equvariant GHA, if f: $X \to Y$ and ϕ: $\Gamma \to G$ are ϵ-GHAs such that

$$d_Y(f(\alpha(x)), \phi(\alpha)(f(x))) < \epsilon, x \in X, \alpha \in \Gamma.$$

Similarly, one defines an equivariant GH-distance,

$$d_{\text{eqGH}}((X, \Gamma), (Y, G)) = \inf\{\epsilon, \text{ there is an } \epsilon - \text{equivariant GHA, } (f, \phi): (X, \Gamma) \to (Y, G)\}$$

and the set of isometric class of (X, Γ) equipped with d_{eqGH} is a complete metric space.

Based on the above, it is not hard to see the following:

Lemma 2.2 Let $X_i \xrightarrow{\text{GH}} X$ be a sequence of compact metric spaces. Assume that each X_i admits a Γ_i-action, Γ_i a closed subgroup of isometries. Then there is a close group G of isometries on X such that passing to a subsequence, $(X_i, \Gamma_i) \xrightarrow{\text{eqGH}} (X, G)$. Moreover, the quotient metric spaces, $X_i/\Gamma_i \xrightarrow{\text{GH}} X/G$.

Consider a sequence of connected complete and locally compact metric spaces $\tilde{X}_i$, and closed subgroup of isometries on $\tilde{X}_i$, Γ_i, such that $\operatorname{diam}(\tilde{X}_i/\Gamma_i) \leqslant d$ for all i. Assume the following commutative diagram of equivariant GH-convergence:

$$\begin{array}{ccc} (\tilde{X}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{\text{eqGH}} & (\tilde{X}, \tilde{p}, G) \\ \downarrow \pi_i & & \downarrow \text{proj} \\ (X_i = \tilde{X}_i/\Gamma_i, p_i) & \xrightarrow{\text{GH}} & (X = \tilde{X}/G, p) \end{array} \quad . \quad (2.1)$$

Lemma 2.3[28] Given X_i in (2.1), assume that G/G_0 is a discrete group, where G_0 denotes the identity component of G. Then for i large, Γ_i contains a normal subgroup Λ_i such that $\Lambda_i \xrightarrow{\text{GH}} G_0$, and Γ_i/Λ_i is isomorphic to G/G_0.

Consider a GH-limit, $M_i \xrightarrow{\text{GH}} X$, where M_i is sequence of compact n-manifolds of $\sec_{M_i} \geqslant -1$ and $\operatorname{diam}_{M_i} \leqslant d$.

Theorem 2.4[30] The isometry group of X is a Lie group. In particular, any closed group of isometries on X is a Lie group.

In our circumstance, to have a commutative diagram of equivariant GH-convergence associated with a sequence, $M_i \xrightarrow{\text{GH}} X$, we need the following property.

Lemma 2.5 (ratio of diameters estimate[28]**)** For $n > 0$, there exists a constant $d(n) > 0$ such that if M is a compact n-manifold of $\operatorname{Ric}_M \geqslant -(n-1)$ and finite fundamental group, then $\dfrac{\operatorname{diam}(\tilde{M})}{\operatorname{diam}(M)} \leqslant d(n)$, a constant depending only on n.

Consider a sequence of compact n-manifolds in lemma 2.2, and $\pi_1(M_i)$ is finite. By lemma 2.3, theorem 2.4 and lemma 2.5, passing to a subsequence, we may assume the following commutative diagram of equivariant GH-convergence, such as

$$\begin{array}{ccc} (\tilde{M}_i, \pi_1(M_i)) & \xrightarrow{\text{eqGH}} & (\tilde{X}, G) \\ \downarrow \pi_i & & \downarrow \text{proj}, \\ M_i & \xrightarrow{\text{GH}} & X = X/G \end{array} \quad (2.2)$$

where the limit group G is a compact Lie group, and π_i: $\tilde{M}_i \to M_i$ denotes the Riemannian universal covering space.

Corollary 2.6[28] Given M_i in (2.2), for i large $\pi_1(M_i)$ contains a normal subgroup Γ_i such that $\Gamma_i \xrightarrow{\text{GH}} G_0$, and $\pi_1(M_i)/\Gamma_i$ is isomorphic to G/G_0, where G_0 denotes the identity component of G.

Note that Γ_i is generated by elements in $\pi_1(M_i)$ whose displacements on $\tilde{M}_i$ are uniformly small.

2.2 Manifolds with bounded sectional curvature and diameter

Consider a sequence of compact n-manifolds, $M_i \xrightarrow{\text{GH}} X$, X is compact, or equivalently $\operatorname{diam}(M_i) \leqslant d$. In this subsection, we will assume that M_i has a uniform bound on sectional curvature, $|\sec_{M_i}| \leqslant 1$. Interplays between geometry and topology of M_i and X are naturally divided into two different categories: noncollapsing i.e., $\operatorname{vol}(M_i) \geqslant v > 0$ for all i; or collapsing i.e., $\operatorname{vol}(M_i) \to 0$ as $i \to \infty$.

The interplays between geometry and topology of M_i and X are well-understood in the next two theorems.

Theorem 2.7 ($C^{1,\alpha}$-convergence, Cheeger-Gromov) Let a sequence of compact n-manifolds, $M_i \to X$, such that

$$|\sec_{M_i}| \leqslant 1, \operatorname{diam}(M_i) \leqslant d, \operatorname{vol}(M_i) \geqslant v > 0.$$

Then X is a smooth manifold such that for i large, there is a diffeomorphism, $f_i: X \to M_i$, such that the f_i-pullback Riemannian metrics on X converges in the $C^{1,\alpha}$-norm.

Corollary 2.8 (diffeomorphism finiteness[31]) Given $n, d, v > 0$, the collection of n-manifolds satisfying

$$\left|\sec_{M_i}\right| \leqslant 1, \operatorname{diam}\left(M_i\right) \leqslant d, \operatorname{vol}\left(M_i\right) \geqslant v > 0,$$

contains at most $c(n, d, v)$-many diffeomorphic types.

Before state a fundamental result on a collapsed manifold with bounded sectional curvature, we need the notion of a singular nilpotent fiber bundles.

Let U be an open subset in a Riemannian n-manifold (e.g., $U = M$), and let $F(U)$ denote the orthogonal frame bundle on U. A singular nilpotent structure on U is defined by a fiber bundle on $F(U)$ (equipped with a canonical metric), $F \to F(U) \xrightarrow{\hat{f}} Y$, with fiber a nil-manifold $F = N/\Gamma$, and affine structural group i.e., a subgroup of $N \rtimes \operatorname{Aut}(N)$, where N is a simply connected nilpotent Lie group, Γ is co-compact lattice, and $\operatorname{Aut}(N)$ is group of automorphisms of N. The $O(n)$-invariance descends the fiber bundle structure on $F(U)$ to a possible singular fibration on $f: U = F(U)/O(n) \to Y/O(n)$; note that a f-fiber has a dimension $<$ dim (F) iff its preimage in $F(U)$ (the $\hat{f}$-fiber) interests with an $O(n)$-orbit a submanifold of positive dimension.

The Gromov's theorem on almost flat manifolds asserts that if $\operatorname{diam}(M_i) < \epsilon(n)$, then M_i is diffeomorphic to an infra-nilmanifold i.e., N_i/Γ_i with N_i a simply connected nilpotent Lie group and Γ_i a discrete subgroup of $N \rtimes \operatorname{Aut}(N)$. The following fundamental structural result may be viewed as a parametrized version of the almost flat manifolds theorem.

Theorem 2.9 (singular nilpotent fibration[32-34]) Let a sequence of compact n-manifolds, $M_i \to X$, such that

$$|\sec_{M_i}| \leqslant 1, \operatorname{diam}(M_i) \leqslant d, \operatorname{vol}(M_i) \to 0.$$

Then for i large, M_i admits a singular nilpotent fibration satisfying the following commutative diagram.

$$\begin{array}{ccc} N_i/\Gamma_i \to (F(M_i), O(n)) & \xrightarrow{\hat{f}_i} & (Y, O(n)) \\ \downarrow \text{proj}_i & & \downarrow \text{proj}, \\ F_i \to M_i & \xrightarrow{\hat{f}_i} & X = Y/O(n) \end{array}$$

and f_i satisfies the following properties:

(p_1) f_i is an ϵ_i-Riemannian submersion and ϵ_i-GHA;

(p_2) The second fundamental form, $|\mathrm{II}_{f_i}| \leqslant c(n)$.

In the proof of theorem 2.9, a technical tool in the proof of theorem 2.9 is to use a nearby metric of higher regularity, obtained via a smoothing method (e.g., Ricci flows, or local embedding to a L^2-space).

2.3 Alexandrov spaces of curvature bounded below

We will review notions and results in Alexandrov geometry that are used in the proof of theorem A.

Alexandrov geometry is a synthetic comparison geometry on a (singular) length metric space on which every triangle satisfies the Toponogov comparison with a comparing triangle on a simply connected surface of constant sectional curvature. Similar to Riemannian curvature comparison geometry, a core issue in Alexandrov geometry is interplays between geometry and topology on an Alexandrov space. Since the introduce of Alexandrov geometry in reference [21], many counterparts to theorems in Riemannian geometry have been established in Alexandrov geometry.

The following Perel'man stability theorem can be treated as a counterpart of theorem 2.7.

Theorem 2.10[22,35] Given a compact Alexandrov n-space X of curvature $\geqslant -1$, there exists $\epsilon(X) > 0$ such that for any Alexandrov n-space Y of curvature $\geqslant -1$, if $d_{\mathrm{GH}}(X, Y) < \epsilon < \epsilon(X)$, then there is a homeomorphism, $f: Y \to X$, which is an ϵ-GHA.

Perel'man claimed that f can be chosen to be a Lipschitz map; though no proof has been available (see reference [35]).

Consider a compact n-manifold M of positive

sectional curvature. The Synge theorem says that if n is even, then up to a double covering M is simply connected (equivalently, any orientation preserving isometry has a fixed point), and if n is odd, M is orientable (equivalently, any orientation reserving isometry has a fixed point). This result easily implies that any isometric T^k-action ($k \geqslant 1$) on the Riemannian universal covering space has a fixed point or a circle T^k-orbit, respectively for n even or odd.

The following is the counterpart of theorem 0.1 in Alexandrov geometry.

Theorem 2.11[23] Let X be a simply connected n-Alexandrov space of curvature $\geqslant 1$. If n is even, then any orientation preserving isometry on X has a fixed point, and if n is odd, any orientation reserving isometry on M has a fixed point.

In the proof of theorem 2.11, the main accomplishment is an establishment of the notion of parallel translation along the interior of a minimal geodesic in X, and a second variation formula in Alexandrov geometry; using which a proof of theorem 2.14 follows essentially the same argument as in the Riemannian case.

Corollary 2.12 Let X be a compact n-Alexandrov space of curvature $\geqslant 1$. If $n = 2m$, then any isometric T^k-action has a fixed point, and if $n = 2m + 1$, then any isometric T^k-action has a circle T^k-orbit.

Proof Assume that $\dim(X) = 2m$. Without loss of the generality, we may assume that X is orientable. Then any $t \in T^k$ is an orientation preserving isometry. By (0.1.2), t has a fixed point x_0. We now choose $t_0 \in T^k$ such that $\langle t_0 \rangle < T^k$ is dense. From $\langle t_0 \rangle (x_0) = x_0$, we conclude that $T^k(x_0) = x_0$.

Assume that $\dim(X) = 2m + 1$. Because corollary 2.12 holds trivially for $k = 1$, we may assume that $k \geqslant 2$.

If the T^k-action has an empty fixed point set, then there is a $T^1 < T^k$ such that T^1-action has no fixed point. Consider the isometric $T^{k-1} = T^k/T^1$-action on X/T^1, an $2m$-dimensional Alexandrov space of positive curvature. Because X is orientable and T^1-action preserves orientation, X/T^1 is orientable. Applying the above result, we conclude that T^{k-1} has a fixed point $\bar{x}_0 \in X/T^1$ i.e., $T^k(x_0) = T^1(x_0)$ is the desired circle orbit.

If the T^k-action on X has a fixed point x_0, we will proceed by induction on $2n + 1$, starting with the trivial case that $n = 1$. Assume that for $\dim(X) = 2n - 1$, the T^k-action has a circle T^k-orbit. For $\dim(X) = 2n + 1$, via isotropy representation one gets an isometric T^k-action on the space of directions at x_0, $\sum_{x_0} X$, which is an $(2n)$-Alexandrov space of curvature $\geqslant 1$. Then the T^k-action on $\sum_{x_0} X$ has a fixed point v, and again via isotropy representation we may consider a T^k-action on $\sum_v \sum_{x_0} X$, an $(2n - 1)$-dimensional Alexandrov space of positive curvature. In any case (whether or not the T^k-action on $\sum_v \sum_{x_0} X$ has a fixed point), we can assume a circle T^k-orbit on $\sum_v \sum_{x_0} X$ (either by the proof of non-fixed point or by applying induction), which implies a circle T^k-orbit near x_0.

In the following, we proof theorem 1.3 by using theorem 2.11.

Proof Assume that $\dim(M) = 2n$. If ϕ is orientation preserving, then ϕ fixes some $p \in M$. If $S^1(p)$ is a circle, then ϕ preserves $S^1(p)$. Otherwise, via differentials, both ϕ and S^1 acts on the unit tangent sphere, thus one can apply induction to complete the proof.

If ϕ is orientation reversing, then ϕ descends to an orientation reversing isometry $\bar{\phi}$ on M/S^1. By theorem 2.11, $\bar{\phi}$ has a fixed point, thus ϕ preserves $S^1(p)$, and as in the above one completes the proof.

Assume $\dim(M) = 2n + 1$. Similarly, one can apply theorem 2.11 to $\bar{\phi}$ on M/S^1 to complete the proof.

3 Proof of theorem 0.7

Lemma 3.1[16] Let M be a compact n-manifold of finite fundamental group.

(3.1.1) If M admits a singular nilpotent fibration, then $\tilde{M}$ admits a $T^k \rtimes \pi_1(M)$-action, where $F = T^k$ (a torus).

(3.1.2) If M admits an almost T^k-invariant metric of $1 \leqslant \sec_M \leqslant K$, then M admits a T^k-invariant metric g of $\frac{1}{2} \leqslant \sec_g \leqslant K$.

Proof (3.1.1) By definition, a singular nilpotent fibration is an $O(n)$-invariant fiber bundle, $F \to (F(M), O(n)) \xrightarrow{\hat{f}} (Y, O(n))$, $F = N/\Gamma$ is a compact nilpotent manifold. Without loss of generality, we may assume that $n \geqslant 3$. Because $\pi_1(M)$ is finite, and $\pi_1(O(n)) \cong Z_2$, $\pi_1(F(M))$ is finite. From the homotopy exact sequence,

$$\cdots \to \pi_2(Y) \to \pi_1(F) \to \pi_1(F(M)) \to \pi_1(Y) \to \cdots,$$

we see that $\pi_1(F)$ has an abelian subgroup of finite index, thus $\pi_1(F)$ is abelian i.e., F is a torus.

(3.1.2) The desired invariant metric can be obtained via Ricci flows to produce a nearby metric of higher regularity, followed by averaging over the almost isometric T^k-actions. By maximal principle, the maximum and minimum of sectional curvature through a Ricci flows change continuously in time, and if an almost T^k-invariant metric satisfies a higher regularity, then averaging which yields an invariant metric whose maximum and minimum remains close. For details, see reference [15].

Proof of (0.7.1) Arguing by contradiction, assuming a sequence of compact n-manifolds M_i such that $1 \leqslant \sec_{M_i} \leqslant K$ and any cyclic subgroup of $\pi_1(M_i)$ has index$\geqslant i, i \to \infty$.

Because $\sec_{M_i} \geqslant 1$, $\operatorname{diam}(M_i) \leqslant \pi$. By corollary 2.8, we may assume that $\operatorname{vol}(M_i) \to 0$. By theorem 0.1 and (3.1.1), for i large we may assume that $\tilde{M}_i$ admits an almost isometric T^{k_i}-action, $k_i \geqslant 1$. By (3.1.2), we may assume that $\tilde{M}_i$ admits an T^{k_i}-invariant metric g_i of $\frac{1}{2} \leqslant \sec_{M_i} \leqslant K$. By now we apply theorem 0.5 to see a contradiction.

A proof of (0.7.2) needs some preparation.

Lemma 3.2[36] Let $(N_i, \Gamma_i) \to (Y, G)$ be a sequence of compact n-manifolds of $\sec_{N_i} \geqslant -1$ and $\operatorname{diam}(N_i) \leqslant d$, and Γ_i are closed group of isometries on N_i. If $\operatorname{vol}(N_i) \geqslant v > 0$, then for i large, there is an ϵ_i-equivariant GHA, $(f_i, \phi_i): (N_i, \Gamma_i) \to (\tilde{Y}, G)$, $\epsilon_i \to 0$, and ϕ_i is an injective homomorphism.

Lemma 3.2 still holds if replacing "$\sec_{M_i} \geqslant -1$" with "$\operatorname{Ric}_{M_i} \geqslant -(n-1)$" (see comments following theorem 2.20 in reference [37]).

Lemma 3.3 Let H and G be closed group of isometries, and let G be a connected Lie group equipped with a left invariant metric of convex radius (at $e \in G$) $\rho > 0$. If $f: H \to G$ is $\frac{\rho}{4} > 0$-homomorphism i.e., for all $\alpha, \beta \in \pi_1(M)$, $d(f(\alpha\beta), f(\alpha) f(\beta)) < \frac{\rho}{4}$, then there is a homomorphism, $\hat{f}: H \to G$, such that $d(f, \hat{f}) < \rho$.

Proof Because the metric on G is left invariant, every point has convex radius ρ. We shall modify $f: H \to G$ to be a homomorphism. For $a \in H$, the set, $\{ f(x)^{-1} f(a)^{-1} f(ax), x \in H \} \subset B_{\frac{\rho}{2}}(e)$, thus has a unique center of mass, $\hat{b}$. Define $\hat{f}: H \to G$, $\hat{f}(a) = \hat{b}$. It is easy to see that $\hat{f}(ab) = \hat{f}(a) \hat{f}(b)$.

Proof of lemma 3.2 We first observe that two metric on X, a bi-invariant metric and a natural metric defined using the distance structure on X are equivalent.

Let $(f_i, \phi_i): (N_i, \Gamma_i) \to (Y, G)$ be ϵ_i-equivariant GHA, $\epsilon_i \to 0$. By lemma 2.5, we may assume a normal subgroup, $\Lambda_i < \Gamma_i$, such that for i large $\Gamma_i/\Lambda_i \cong G/G_0$.

By lemma 3.3, for i large we may assume a homomorphism $\Phi_i: \Lambda_i \to G_0$. We claim that Φ_i is injective. Arguing by contradiction, assuming $e \neq t_i \in \Lambda_i$, $\Phi_i(t_i) = e$. Then

$$\begin{array}{ccc} N_i & \longrightarrow & Y \\ \downarrow \text{proj}_i & & \downarrow \text{id}_Y \\ N_i/\langle \gamma_i \rangle & \xrightarrow{\text{GH}} & Y/e \end{array}.$$

A contradiction to the volume convergence, because both $\operatorname{vol}(N_i)$ and $\operatorname{vol}(N_i/\langle \gamma_i \rangle) \leqslant \frac{\operatorname{vol}(N_i)}{2}$ converges to $\operatorname{vol}(Y) = \operatorname{vol}(Y/\{e\})$.

Let's point it out a fact on lemma 3.3: Γ_i acts freely and discontinuously on $\tilde{M}_i$, but $\phi_i(\Gamma_i)$ may not act freely on $\tilde{X}$, see the following example.

Example 3.4[38] There is a family of metrics on $S^3\times S^3, g_i$, such that $\mathrm{Ric}_{g_i}\geqslant 5$ and $\mathrm{diam}(g_i)\to\pi$, with a free isometric S^1-action, and $\left(S^3\times S^3, g_i\right)\overset{\mathrm{GH}}{\to}\tilde{X}$, where $\tilde{X}$ is a spherical suspension over $S^3\times S^2$ equipped with a metric of diameter$<\pi$. Let $\mathbb{Z}_i<S^1$, then

$$\begin{array}{ccc}(S^3\times S^3, g_i, \mathbb{Z}_i) & \xrightarrow{\mathrm{eqGH}} & (X, S^1)\\ \downarrow \pi_i & & \downarrow \mathrm{proj}.\\ (S^3\times S^3)/\mathbb{Z}_i & \xrightarrow{\mathrm{GH}} & X=\tilde{X}/S^1\end{array}$$

Note that $\phi_i:\mathbb{Z}_i\to S^1$ in lemma 3.3 can be chosen to be the natural inclusion. Because the S^1-action fixes the two vertices on $\tilde{X}$, $\phi_i(\mathbb{Z}_i)$ fixes the two vertices.

The following is the key property in the proof of (0.7.2).

Lemma 3.5 Let $(f_i,\phi_i):(\tilde{M}_i,\Gamma_i)\to(\tilde{X},T^k)$ be as in lemma 3.2. Suppose that $\sec_{M_i}\geqslant -1$. Then for i large, $\phi_i(\Gamma_i)$ acts freely on $\tilde{X}$ In particular, the T^k-action has no fixed point in $\tilde{X}$.

Remark 3.6 (3.6.1) As seen in example 3.4, lemma 3.5 is false if one weakens the curvature condition, "$\sec_{M_i}\geqslant -1$", to "$\mathrm{Ric}_{M_i}\geqslant -(n-1)$". However, the proof of lemma 3.5 remains valid with $\mathrm{Ric}_{M_i}\geqslant -(n-1)$ and the additional condition: all points in $\tilde{M}_i$ are (δ, n)-Reifenberg points, $0<\delta<\delta(n)$, i.e.there is $\rho>0$ $d_{\mathrm{GH}}(B_s(\tilde{x}_i), \underline{B}^n_s(0))<s\cdot\delta$ for all $0<s\leqslant\rho$, where $\underline{B}^n_s(0)$ denotes a s-ball in the Euclidean n-space.

(3.6.2) Based on the precompactness result in reference [39], the proof of lemma 3.5 (with an obvious minor modification) implies the following local property: given a sequence of collapsed n-manifold M_i of $\sec_{M_i}\geqslant -1$ and $x_i\in M_i$ such that

$$\begin{array}{ccc}(\pi_i^{-1}(B_\rho(x_i))_{\tilde{x}_i}/\Lambda_i, \tilde{x}_i, \Gamma_i) & \xrightarrow{\mathrm{eqGH}} & (\tilde{X}, \tilde{x}, G)\\ \downarrow \pi_i & & \downarrow \mathrm{proj}\\ (B_\rho(x_i), x_i) & \xrightarrow{\mathrm{GH}} & B_\rho(x)=\tilde{X}/G\end{array},$$

where $\pi_i:(\widetilde{B_{2\rho}(x_i)}, \tilde{x}_i)\to(B_{2\rho}(x_i), x_i)$ denotes the Riemannian universal cover, $\pi_i^{-1}(B_\rho(x_i))_{\tilde{x}_i}$ denotes the component at $\tilde{x}_i$, $\Gamma_i=\mathrm{im}(\iota_*)$, $\Lambda_i=\ker(\iota_*)$, $\iota_*:\pi_1(B_\rho(x_i))\to\pi_1(B_{2\rho}(x_i), x_i)$, and $\iota: B_\rho(x_i)\hookrightarrow B_{2\rho}(x_i)$ denotes the inclusion map. Assume that for all i, $\mathrm{vol}(B_\rho(\tilde{x}_i))\geqslant v>0$. Then G_0 has no fixed point.

Proof of lemma 3.5 Arguing by contradiction, there is a sequence of ϵ_i-equvairant GHA, (f_i,ϕ_i): $(\tilde{M}_i,\Gamma_i)\to(\tilde{X},T^k)$, satisfying lemma 3.5, and for each i, there are $e\neq\gamma_i\in\Gamma_i$ and $\tilde{z}_i\in\tilde{X}$ such that $\phi_i(\gamma_i)(\tilde{z}_i)=\tilde{z}_i$. Because $\tilde{X}$ is compact ($\mathrm{diam}(\tilde{M})\leqslant\pi$), passing to a subsequence we may assume that $\langle\gamma_i\rangle\to H<T^k$ and $\tilde{z}_i\to\tilde{z}_0\in\tilde{X}$.

Because ϕ_i is an injective homomorphism (lemma 3.5), for any $m\geqslant 1$,

$$\phi_i(\gamma_i^m)(\tilde{z}_i)=(\phi_i(\gamma_i))^m(\tilde{z}_i)=$$
$$(\phi_i(\gamma_i))^{m-1}(\phi_i(\gamma_i)(\tilde{z}_i))=(\phi_i(\gamma_i))^{m-1}(\tilde{z}_i)=\cdots=\tilde{z}_i,$$
$$\mathrm{d}_{\tilde{X}}\left(\phi_i\left(\gamma_i^m\right)\left(\tilde{z}_0\right),\tilde{z}_0\right)\leqslant$$
$$\mathrm{d}_{\tilde{X}}\left(\phi_i\left(\gamma_i^m\right)\left(\tilde{z}_i\right),\tilde{z}_0\right)+\mathrm{d}_{\tilde{X}}\left(\phi_i\left(\gamma_i^m\right)\left(\tilde{z}_i\right),\phi_i\left(\gamma_i^m\right)\left(\tilde{z}_0\right)\right)=$$
$$\mathrm{d}_{\tilde{X}}\left(\tilde{z}_i,\tilde{z}_0\right)+\mathrm{d}_{\tilde{X}}\left(\tilde{z}_i,\tilde{z}_0\right)=2\mathrm{d}_{\tilde{X}}\left(\tilde{z}_i,\tilde{z}_0\right)\to 0.$$

Consequently,

$$H(\tilde{z}_0)=\tilde{z}_0. \qquad (3.5.1)$$

Consider the following equivariant convergence,

$$\begin{array}{ccc}(\tilde{M}_i,\langle\gamma_i\rangle<\Gamma_i) & \xrightarrow{(f_i,\ \phi_i)} & (\tilde{X}, H<T^k)\\ \downarrow \hat{\pi}_i & & \downarrow \mathrm{proj}\\ \hat{M}_i=\tilde{M}_i/\langle\gamma_i\rangle & \xrightarrow{\mathrm{GH}} & \hat{X}=\tilde{X}/H\end{array}, \qquad (3.5.2)$$

where $\{e\}<H$ (note that it is allowed that H is finite, or $H=T^k$).

Without loss of the generality, we may assume that γ_i is chosen so that $o(\gamma_i)=p_i$, a prime. We shall show that $\langle\gamma_i\rangle$ has a fixed point, a contradiction. Moreover, we will present two different proofs.

(i) Topological method. Because $\tilde{X}=f_i(\tilde{M}_i)$ is a topological manifold, we claim that there is a small $\eta>0$ such that for i large, the following properties hold:

(3.5.3) $B_\eta(\tilde{z}_0)$ is homeomorphic to an open Euclidean ball.

(3.5.4) Let $\tilde{U}_i=f_i^{-1}(B_{\frac{9}{10}\eta}(\tilde{z}_0))$, and let $\tilde{V}_i=\langle\gamma_i\rangle(\tilde{U}_i)=\bigcup_{k=1}^{p_i}\gamma_i^k(\tilde{U}_i)$. For i large, $B_{\frac{4}{5}\eta}(\tilde{z}_0)\subset f_i(\tilde{V}_i)\subset B_\eta(\tilde{z}_0)$.

(3.5.5) There is an isotopy which is a small perturbation of $\mathrm{id}_{B_\eta(\tilde z_0)}$, $\psi_t: B_\eta(\tilde z_0)\to B_\eta(\tilde z_0)$, $\psi_0=\mathrm{id}_{B_\eta(\tilde z_0)}$, $\psi_1(B_\eta(\tilde z_0))=B_{\frac{4}{5}\eta}(\tilde z_0)$, with gradient field $\nabla\psi_t$ transversal to $\partial(f_i(\tilde V_i))$.

By (3.5.3~3.5.5), one concludes that $\tilde V_i$ is homeomorphic to a Euclidean ball. Because $\tilde V_i$ is $\langle\gamma_i\rangle$-invariant, applying Smith fixed point theorem to $(\tilde V_i, \langle\gamma_i\rangle\cong \mathrm{Z}_{p_i})$, we conclude $\langle\gamma_i\rangle$-action has a fixed point in $\tilde V_i$, a contradiction.

To see the claim, consider a blow up limit at $\tilde z_0$, $(\epsilon_i^{-\frac12}\tilde X,\tilde z_0,H)\xrightarrow{\mathrm{GH}}(C(\Sigma_{\tilde z_0}),o,\mathrm{d}H)$, where the tangent cone, $C(\Sigma_{\tilde z_0})$, is a metric cone over the space of directions at $\tilde z_0$, $\Sigma_{\tilde z_0}$, which is an Alexandrov space of curv $\geqslant 1$ and homeomorphic to a sphere[40], and the limit group dH may be treated as the 'differential' of the H-action such that $\mathrm{d}H(o)=o$. We may assume a $4\epsilon_i^{\frac12}$-equivariant GHA, $(\tilde f_i,\tilde\phi_i):(\epsilon_i^{-\frac12}\tilde M_i,\langle\gamma_i\rangle)\to(C(\Sigma_{\tilde z_0}),dH)$, where $\tilde f_i=h_i\circ f_i$, $h_i:(\epsilon_i^{-\frac12}\tilde M_i,\tilde x_i)\to(C(\Sigma_{\tilde z_i}),o)$ is a homeomorphic embedding and $\epsilon_i^{-\frac12}$-GHA on $B_{\frac14\epsilon_i^{-\frac12}}(\tilde x_i,\epsilon_i^{-\frac12}\tilde M_i)\to C(\Sigma_{\tilde z_0})$, $h_i(\tilde x_i)=o$, and $\tilde\phi_i(\gamma_i)=d(\phi_i(\gamma_i))$. Fixing $\eta>1$, it is clear that for i large (3.5.3~3.5.5) hold for $(\tilde f_i,\tilde\phi_i):(\epsilon_i^{-\frac12}\tilde M_i,\langle\gamma_i\rangle)\to(C(\Sigma_{\tilde z_0}),\mathrm{d}H)$, which implies that (3.5.3~3.5.5) hold for $(f_i,\phi_i):(\tilde M_i,\langle\gamma_i\rangle)\to(\tilde X,H)$.

(ii) Geometrical method. By reference [41-42], we may assume a convex neighborhood $\tilde U\supset B_\rho(\tilde z_0)$ $(\rho>0)$ defined by a (local) convex function approximating $\frac12 d(\tilde z_0,\cdot)^2$ up to a second order, and for i large the convex function lifts to a function at $\tilde x_i\in\tilde M_i$ such that the lifting function is convex and determines a convex neighborhood which contains $B_{\frac{\rho}{2}}(\tilde x_i)$. Consider the energy functional

$$E(\tilde x_i,\cdot)=\frac12\sum_{k=0}^{p_i}\mathrm{d}_i^2\left(\gamma_i^k(\tilde x_i),\cdot\right):B_1\left(f_i^{-1}(\tilde U)\right)\to\mathbb{R}_+.$$

Because $\langle\gamma_i\rangle(\tilde x_i)\to H(\tilde z_0)=\tilde z_0$, for large we may assume that $E(\tilde x_i,\cdot)$ achieves a unique minimum in $B_{\frac{\rho}{2}}(\tilde x_i)$, which is fixed by $\langle\gamma_i\rangle$ (a contradiction).

Proof of (0.7.2) Arguing by contradiction, we may assume a sequence of compact n-manifolds (n is odd), $M_i\xrightarrow{\mathrm{GH}}X$, satisfying that $\sec_{M_i}\geqslant 1$, $\mathrm{vol}(\tilde M_i)\geqslant v>0$, and any cyclic subgroup of $\pi_1(M_i)$ has index$\geqslant i$. We will derive a contradiction by showing that for i large, $\pi_1(M_i)$ contains a cyclic subgroup of finite bounded by a finite constant.

Passing to a subsequence, we obtain the commutative diagram in (2.1): for i large we may assume that $\Gamma_i=\pi_1(M_i)$ has a normal subgroup Λ_i such that $\pi_1(M_i)/\Gamma_i\cong G/G_0$ (lemma 2.5). Without loss of generality, we may assume that $\mathrm{vol}(M_i)\to 0$, thus $G_0\neq\{e\}$. Because $|\Gamma_i/\Lambda_i|=|G/G_0|<\infty$, we will derive a contradiction by showing that for i large, Λ_i is cyclic.

By lemma 3.2, for i large we may assume an injective homomorphism $\phi_i:\Lambda_i\to G_0$. Recall the following compact version of Margulis lemma (theorem A in reference [42]: if H is a connected compact Lie group, then a discrete subgroup of H which is generated by short elements is abelian). Then $\phi_i(\Lambda_i)$ is abelian. Because $\psi_i(\Lambda_i)\to G_0$, $G_0=T^k$. Without loss of generality, we may assume that $k\geqslant 2$. By corollary 2.12, the T^k-action has a circle T^k-orbit. By lemma 3.5, $\phi_i(\Lambda_i)$ acts freely on the circle T^k-orbit, thus $\Lambda_i\cong\phi_i(\Lambda_i)$ is cyclic.

Acknowledgement The author would like to thank Pan Jiayin for a useful comment on an early draft.

©The editorial department of Journal of Capital Normal University (Natural Science Edition) 2024. This is an open access article under the CC BY-NC-ND 4.0 license (https://creativecommons.org/licenses/by-nc-nd/4.0/).

References

[1] KAPOVITCH V, WILKING B. Structure of fundamental groups of manifolds with Ricci curvature bounded below [PP/OL]. [2024-07-02]. https://arxiv.org/pdf/1105.5955.

[2] WEI G. Examples of complete manifolds of positive Ricci curvature with nilpotent isometry group [J]. Bulletin of the American Mathematical Society, 1988,

19:311-313. DOI:10.1090/S0273-0979-1988-15653-4.

[3] WILKING B. On fundamental groups of manifolds of nonnegative curvature[J]. Differential Geometry and its Applications, 2000, 13: 129-165. DOI: 10.1016/S0926-2245(00)00030-9.

[4] BRUÉ B, NABER A, SEMOLA D. Fundamental groups and the Milnor conjecture [PP/OL]. [2024-06-28]. https://arxiv.org/pdf/2303.15347.

[5] BRUÉ B, NABER A, SEMOLA D. Six dimensional counterexample to the Milnor conjecture [PP/OL]. [2024-06-28].https://arxiv.org/abs/2311.12155.

[6] BRUÉ B, NABER A, SEMOLA D. Compact manifolds with unbounded nilpotent fundamental groups and positive Ricci curvature[PP/OL]. [2024-06-28]. https://doi.org/10.48550/arXiv.2504.09603.

[7] MYERS S B. Riemannian manifolds with positive mean curvature[J]. Duke Mathematical Journal, 1941, 8: 401-404. DOI:10.1215/S0012-7094-41-00832-3.

[8] CHEEGER J, GROMOLL D. The splitting theorem for manifolds of nonnegative Ricci curvature[J]. Journal of Differential Geometry, 1971, 6: 119-128. DOI: 10.4310/jdg/1214430220.

[9] GROMOLL D, MEYER W. On complete open manifolds of positive curvature [J]. Annals of Mathematics, 1969, 90(1): 75-90.

[10] SYNGE J L. On the connectivity of spaces of positive curvature[J]. Quarterly Journal of Mathematics, 1936, os-7(1): 316-320. DOI:10.1093/qmath/os-7.1.316.

[11] WOLF J A. Spaces of constant curvature[M]. 6th ed. Washington: American Mathematical Society, 2015.

[12] CHERN S S. Proceedings of the United States-Japan seminar in differential geometry [M]. Kyoto: Japan, 1965.

[13] SHI W X. Deforming the metric on complete Riemannian manifolds [J]. Journal of Differential Geometry, 1989, 30 (1): 223-301. DOI: 10.4310/jdg/1214443292.

[14] GROVE K, SHANKAR K. Rank two fundamental groups of positively curved manifolds[J]. The Journal of Geometric Analysis, 2000, 10 (4): 679-682. DOI: 10.1007/BF02921991.

[15] BAZAĬKIN Y V. A manifold with positive sectional curvature and fundamental group $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ [J]. Siberian Mathematical Journal, 1999, 40(5): 994-996.

[16] RONG X C. On the fundamental group of manifolds of positive sectional curvature[J]. Annals of Mathematics, 1996, 143(2): 397-411.

[17] RONG X C. Positive curvature, local and global symmetry, and fundamental groups [J]. American Journal of Mathematics, 1999, 121(5): 931-943. DOI: 10.1353/ajm.1999.0036.

[18] RONG X C. Positively curved manifolds with Abelian symmetry [C]//International Congress of Chinese Mathematicians. The Fourth International Congress of Chinese Mathematicians. Hangzhou: Zhejiang University, 2007.

[19] SU X L, WANG Y S. On finite subgroups of compact Lie groups and fundamental groups of Riemannian manifolds[J]. Advances in Geometry, 2011, 11(2): 191-199. DOI: 10.1515/advgeom.2011.001.

[20] RONG X C. Fundametal groups of manifolds of positive sectional curvature and bounded covering geometry[J]. Journal of Mathematical Study, 2024, 57(3): 358-372. DOI: 10.4208/jms.v57n3.24.07.

[21] BURAGO Y, GROMOV M, PEREL'MAN G A D. Alexandov spaces with curvature bounded below [J]. Uspekhi Matematicheskikh Nauk, 1992, 47(2): 3-51.

[22] BURAGO Y D, GROMOV M L, PEREL'MAN G A D. Alexandrov spaces with curvatures bounded from below [J]. Russian Mathematical Surveys, 1992, 47(2): 1-58. DOI: 10.1070/RM1992v047n02ABEH000877.

[23] PETRUNIN A. Parallel transportation for Alexandrov space with curvature bounded below[J]. Geometric and Functional Analysis, 1998, 8: 123-148. DOI: 10.1007/s000390050050.

[24] BROWN K S. Cohomology of groups[M]. New York: Springer-Verlag, 1982.

[25] BREDON G. Introduction to compact transformation groups[M]. New York: Academic Press, 1972.

[26] GROMOV M. Curvature diameter and Betti numbers [J]. Commentarii Mathematici Helvetici, 1981, 56: 179-195. DOI: 10.1007/BF02566208.

[27] KAPOVITCH V, PETRUNIN A, TUSCHMANN W. Nilpotency, almost nonnegative curvature, and the gradient flow on Alexandrov spaces [J]. Annals of Mathematics, 2010, 171 (1): 343-373. DOI: 10.4007/annals.2010.171.343.

[28] FUKAYA K, YAMAGUCHI T. The fundamental groups of almost non-negatively curved manifolds[J]. Annals of Mathematics, 1992, 136(2): 253-333. DOI: 10.2307/2946606.

(下转第27页)

[14] LUO P. Blow-up phenomena for a pseudo-parabolic equation [J]. Mathematical Methods in the Applied Sciences, 2015, 38(12): 2636-2641. DOI: 10.1002/mma.3253.

[15] PENG X M, SHANG Y D, ZHENG Z Z. Blow-up phenomena for some nonlinear pseudo-parabolic equations, [J] Applied Mathematics Letters, 2016, 56: 17-22. DOI:10.1016/j.aml.2015.12.005.

[16] LI N, FANG S M. Finite time blow-up with upper bound of blowup time of solutions to semilinear parabolic equations with general nonlinearity [J]. Chinese Quarterly Journal of Mathematics, 2022, 37(1): 103-110. DOI:10.13371/j.cnki.chin.q.j.m.2022.01.011.

[17] LEVINE H. Some nonexistence and instability theorems or solutions of formally parabolic equation of the form $Pu_t = -Au + \mathcal{F}(u)$[J]. Archive for Rational Mechanics and Analysis, 1973, 51: 371-386. DOI: 10.1007/BF00263041.

[18] LEVINE H. Some additional remarks on the nonexistence of global solutions to nonlinear-wave equations [J]. Siam Journal on Mathematical Analysis, 1974, 5(1): 138-146. DOI:10.1137/0505015.

[19] SUN F L, LIU L S, WU Y H. Finite time blow-up for a class of parabolic or pseudo-parabolic equations [J]. Computers & Mathematics with Applications, 2018, 75(10): 3685-3701. DOI:10.1016/j.camwa.2018.02.025.

[20] LAWRENCE C E. Partial differential equations [M]. New York: America Mathematical Society, 2010.

(责任编辑:马田田)

(上接第13页)

[29] RONG X C. Notes on convergence and collapsing theorems in Riemannian geometry [J]. Handbook of Geometric Analysis, 2010, II(13): 193-298.

[30] FUKAYA K, YAMAGUCHI T. Isometry groups of singular spaces [J]. Mathematische Zeitschrift, 1994, 216: 31-44. DOI: 10.1007/BF02572307.

[31] CHEEGER J. Finiteness theorems for Riemannian manifolds [J]. American Journal of Mathematics, 1970, 92: 61-75.

[32] FUKAYA K. Collapsing Riemannian manifolds to ones with lower dimensions [J]. Journal of Differential Geometry, 1987, 125: 139-156.

[33] FUKAYA K. A boundary of the set of Riemannian manifolds with bounded curvature and diameters [J]. Journal of Differential Geometry, 1988, 28: 1-21.

[34] CHEEGER J, FUKAYA K, GROMOV M. Nilpotent structures and invariant metrics on collapsed manifolds [J]. Journal of the American Mathematical Society, 1992, 5(2): 327-372. DOI: 10.2307/2152771.

[35] KAPOVITCH V. Perelman's stability theorem [PP/OL]. [2024-06-21].https://arxiv.org/abs/math/0703002.

[36] MASUR M, RONG X C, WANG Y S. Margulis lemma for compact lie groups [J]. Mathematische Zeitschrift, 2008, 258: 395-406. DOI: 10.1007/s00209-007-0178-4.

[37] PAN J Y, RONG X C. Ricci curvature and isometric actions with scaling nonvanishing property [PP/OL]. [2024-06-28].https://arxiv.org/abs/1808.02329.

[38] OTSU Y. On manifolds of positive Ricci curvature with large diameter [J]. Mathematische Zeitschrift, 1991, 206: 255-264. DOI: 10.1007/BF02571341.

[39] XU S C. Precompactness of domains with lower Ricci curvature bound under Gromov-Hausdorff topology [PP/OL]. [2024-06-28]. https://arxiv.org/abs/2311.05140.

[40] KAPOVITCH V. Regularity of limits of noncollapsing sequences of manifolds [J]. Geometric and Functional Analysis, 2002, 12: 121-137. DOI: 10.1007/s00039-002-8240-1.

[41] NEPECHIY A. Toward canonical convex functions in Alexandrov spaces [PP/OL]. [2024-06-28]. https://arxiv.org/abs/1910.00253.

[42] LI N. Quantitative estimates on the C^2-singular sets in Alexandrov spaces [PP/OL]. [2024-06-28]. https://api.semanticscholar.org/CorpusID:259088969.

(责任编辑:马田田)