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# 一类半线性伪抛物方程爆破时间的上下界估计

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**摘要**:为了研究一类半线性伪抛物方程解的爆破行,给出模型在不同初始情况下的爆破时间上下界估计。本文利用微分不等式技巧得到了在低初始能量下爆破时间的下界估计,得到了在高初始能量下的爆破时间下界估计,并利用凸方法得到了在任意初始能量下爆破时间的上界估计。

**关键词**:半线性伪抛物方程;解爆破时间上下界的估计;微分不等式法;凸方法
**中图分类号**:O29 **文献标志码**:A

# Upper and lower bounds of blow-up time for a class of semi-linear pseudo-parabolic equation

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**Abstract**: In this paper, the blow-up phenomena of a class of semi-linear pseudo-parabolic equations are studied, the estimation of upper and lower bounds of blow-up time under different initial conditions are given. To be specific, the lower bound of blow-up time under low initial energy is obtained by the means of using differential inequality technique, and the lower bound of blow-up time under high initial energy is obtained similarly. Then, the upper bound of blow-up time under any initial energy is obtained by the means of using convex method.

**Keywords**: semi-linear pseudo-parabolic equation; upper and lower bounds of blow-up time; differential inequality method; convex method
**CLC**:O29 **DC**:A

## 0 引 言

本文研究半线性伪抛物方程

$$\begin{cases} u_t - \Delta u - \Delta u_t = u^p - u, (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0, x \in \Omega \end{cases} \quad (M1)$$

的初边值问题,式中:$\Omega \subset \mathbf{R}^n$是一个具有光滑边界$\partial\Omega$的有界区域;$T$表示解的最大存在时间,可以为有限数或无穷大。指标 $p$ 满足

$$p = \begin{cases} (1, +\infty), & n = 1, 2, \\ \left(1, \dfrac{n+2}{n-2}\right), & n \geqslant 3。 \end{cases} \quad (M2)$$

伪抛物方程常描述涉及双温度的热传导,非线性色散长波的单向传播和种群聚集等物理现象,具有很强的实际应用背景和深刻的理论价值[1-3]。

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对于模型(M1)的爆破标准,已经有很多学者进行研究。Emmanuele和Michel[4]得到了模型(M1)的比较原理;Xu和Su[5]利用位势井理论[6-8]在低初始能量下讨论了解的全局存在性问题;Zhu等[9]得到了解在有限时间爆破的充分必要条件,给出定义:

$$I(u) \coloneqq \|u\|_{H_0^1}^2 - \|u\|_{p+1}^{p+1},$$

$$\mathcal{N}_- = \{ u \in H_0^1(\Omega) : I(u) < 0 \},$$

$$S_- = \{ u_0 \in H_0^1(\Omega) : \forall t_0 \in [0, T),\ u(t_0) \in \mathcal{N}_- \}。$$

**引理1**[9] 假设 $u_0 \in H_0^1(\Omega)$,则模型(M1)的解在有限时间爆破当且仅当存在 $t_0 \in [0, T)$,使得 $I\left(u(t_0)\right) < 0$。此外,若模型(M1)所考虑的区域 $\Omega \subset \mathbf{R}^n$ 具有光滑边界,此时集合 $S_- \subsetneq \mathcal{N}_-$。

基于引理1以及Xu和Su[5]的相关结果,对模型(M1)的爆破情况进行如下归纳:

$I(u_0) < 0$,模型(M1)的解在有限时刻爆破;

$J(u_0) \leqslant 0, I(u_0) > 0$,2个条件矛盾;

$0 < J(u_0) \leqslant d, I(u_0) > 0$,模型(M1)的解全局存在;

$J(u_0) > d, I(u_0) > 0$,若存在 $t_0 \in (0, T]$,满足 $I\left(u(t_0)\right) < 0$,模型(M1)的解在有限时刻爆破。

上述结果表明模型(M1)已经获得解的爆破准则,但是对其解的爆破行为还没有清楚的刻画。对于解的爆破时间的估计,国内外众多学者致力于通过不同方法获得各类模型解的爆破时间估计[10-13]。因此,为了得到模型解的爆破行为,特别是解的爆破时间的估计,本文基于以上爆破准则,主要研究模型(M1)的解在不同初始能量情况发生爆破时,解的爆破时间上下界估计。利用微分不等式法[14-15]给出当初值满足 $J(u_0) < 0, I(u_0) < 0$ 时的一个爆破时间下界估计;结合位势井中的结果得到当初值满足 $J(u_0) < d, I(u_0) < 0$ 时爆破时间下界估计,然后利用凸方法[16-19]得到当初值满足 $I(u_0) < 0$ 时爆破时间上界估计。

## 1 预备知识

本文有如下规定:

$\|u\|_q$:勒贝格空间 $L^q(\Omega)$, $q \in [1, +\infty]$ 上的范数。

$\|u\|_{H_0^1(\Omega)}$:索伯列夫空间 $H_0^1(\Omega)$ 上的范数。

定义泛函

$$I(u) \coloneqq \|u\|_{H_0^1}^2 - \|u\|_{p+1}^{p+1}。 \tag{1}$$

定义能量泛函

$$J(u) \coloneqq \frac{1}{2}\|u\|_{H_0^1}^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}。 \tag{2}$$

定义位势井深度

$$d = \inf_{u \in \mathcal{N}} J(u), \tag{3}$$

其中 $\mathcal{N} = \{ u \in H_0^1(\Omega) | I(u) = 0,\ \|u\|_{H_0^1} \neq 0 \}$。

根据Rellich-Kondrachov紧嵌入定理[20]记

$$C = \sup_{u \in \mathcal{N}} \frac{\|u\|_{p+1}}{\|u\|_{H_0^1}}。 \tag{4}$$

考虑 $-\Delta$ 的特征值问题,其最小特征值又被称为第一特征值,记为 $\lambda_1$。

**定义2** 若定义在 $\Omega \times [0, T)$ 上的函数 $u = u(x, t)$ 满足 $u \in L^\infty\left(0, T; H_0^1(\Omega)\right)$, $u_t \in L^2\left(0, T; H_0^1(\Omega)\right)$。$u_0(x) \in H_0^1(\Omega)$ 还满足条件:

(1)任意 $v \in H_0^1(\Omega), t \in (0, T)$,

$$\int_\Omega u_t v + \nabla u_t \cdot \nabla v + \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega u^p v - uv \mathrm{d}x; \tag{5}$$

(2) $u(x, t) = 0, (x, t) \in \partial\Omega \times (0, T)$;

(3)任意 $t \in [0, T)$, $\int_0^t \|u_\tau\|_{H_0^1}^2 \mathrm{d}\tau + J\left(u(t)\right) \leqslant J(u_0)$;

则称 $u$ 为模型(M1)的弱解。

**定义3** $u(t)$ 是模型(M1)的弱解,称 $u(t)$ 的最大存在时间为 $T$。如果 $T$ 有限且满足

$$\lim_{t \to T} \int_0^T \|u\|_{H_0^1}^2 \mathrm{d}\tau = +\infty, \tag{6}$$

则称 $u(t)$ 在有限时间爆破。

**引理4**[9] 位势井深度 $d$ 满足

$$d = \frac{p-1}{2(p+1)}\left(\frac{1}{C}\right)^{\frac{2(p+1)}{p-1}}。 \tag{7}$$

**引理5** 对 $u \in H_0^1(\Omega)$,若 $I(u_0) < 0$,那么对于任意的 $t \in [0, T)$,满足 $I(u) < 0$。

**证明** 若引理不成立,则根据 $I(u(t))$ 的连续性,存在 $t_0 \in (0, T)$,使得

$$I\left(u(t_0)\right) = 0, I\left(u(t)\right) < 0, \forall t \in [0, t_0)。$$

由于

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H_0^1}^2 = -2\|u\|_{H_0^1}^2 + 2\|u\|_{p+1}^{p+1} = -2I(u),$$

因此对于任意 $t \in [0, t_0)$,满足 $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H_0^1}^2 > 0$。这说明 $\|u\|_{H_0^1}^2$ 在 $[0, t_0)$ 上严格单调递增。因此有

$$J(u_0)=\frac{1}{2}\|u_0\|_{H_0^1}^2-\frac{1}{p+1}\|u_0\|_{p+1}^{p+1}=\frac{I(u_0)}{p+1}+\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_0\|_{H_0^1}^2<\frac{p-1}{2(p+1)}\|u_0\|_{H_0^1}^2<\frac{p-1}{2(p+1)}\|u_0\|_{H_0^1}^2。$$

即

$$J(u_0)<\frac{p-1}{2(p+1)}\|u(t_0)\|_{H_0^1}^2。\tag{8}$$

模型(M1)两边同时点乘 $u_t$ 且关于空间积分,经过简单计算可知

$$\frac{\mathrm{d}}{\mathrm{d}t}J(u)=-||u_t||_{H_0^1}^2\leqslant 0,$$

即 $J(u(t))$ 关于 $t$ 单调递减。因此有

$$J(u_0)>J(u(t_0))=\frac{I(u(t_0))}{p+1}+\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u(t_0)\|_{H_0^1}^2=\frac{p-1}{2(p+1)}\|u(t_0)\|_{H_0^1}^2。$$

即

$$J(u_0)>\frac{p-1}{2(p+1)}\|u(t_0)\|_{H_0^1}^2。\tag{9}$$

式(8和9)矛盾,这说明对于任意的 $t\in[0,T)$,满足 $I(u)<0$。

**引理6**[16] 对于一个正的二阶可微函数 $M(t)$,若存在 $\theta>0$,满足

$$M''(t)M(t)-(1+\theta)(M'(t))^2\geqslant 0\ (t>0),$$

且 $M(0)>0, M'(0)>0$,则存在

$$t\to t_*<t^*=\frac{M(0)}{\theta M'(0)}$$

使得当 $t\to t_*$ 时,$M(t)\to\infty$。

## 2 低初始能量的爆破时间下界估计

接下来给出低初始能量情况下解的爆破时间下界估计以及主要推导过程。

**定理7** 对于模型(M1),$u_0\in H_0^1(\Omega)$ 且满足 $J(u_0)<0$,那么模型(M1)的解在有限时刻 $T$ 处爆破且有下界估计

$$T\geqslant\int_{||u_0||_{H_0^1}^2}^{\infty}\frac{1}{-\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}+4C^{2(p+1)}\phi(t)^p}\mathrm{d}\phi,$$

式中 $\phi(t)=||u(\cdot,t)||_{H_0^1}^2$。

**证明** 定义泛函 $\Phi(t)=||u(\cdot,t)||_{H_0^1}^2$。根据 $J(u)$ 的定义,显然有 $\frac{\mathrm{d}}{\mathrm{d}t}J(u)=-||u_t||_{H_0^1}^2\leqslant 0$,即对 $\forall t\in[0,T)$ 满足 $J(u)<0$。因此

$$||u||_{H_0^1}^2<\frac{2}{p+1}||u||_{p+1}^{p+1}\leqslant\frac{2C^{p+1}}{p+1}||u||_{H_0^1}^{p+1},$$

即

$$\phi(t)>\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}。\tag{10}$$

利用Young不等式,对任意的 $\epsilon>0$ 有

$$\phi'(t)=\frac{\mathrm{d}}{\mathrm{d}t}||u||_{H_0^1}^2=-2I(u)=-2\phi(t)+2||u||_{p+1}^{p+1}\leqslant-2\phi(t)+2C^{p+1}||u||_{H_0^1}^{p+1}=-2\phi(t)+2C^{p+1}\left(||u||_{H_0^1}^{p}||u||_{H_0^1}^{1}\right)\leqslant-2\phi(t)+2C^{p+1}\epsilon||u||_{H_0^1}^2+\frac{2C^{p+1}}{\epsilon}||u||_{H_0^1}^{2p}=\left(-2+2C^{p+1}\epsilon\right)\phi(t)+\frac{2C^{p+1}}{\epsilon}||u||_{H_0^1}^{2p}。$$

即

$$\phi'(t)\leqslant\left(-2+2C^{p+1}\epsilon\right)\phi(t)+\frac{2C^{p+1}}{\epsilon}||u||_{H_0^1}^{2p}。\tag{11}$$

特别地,取 $\epsilon=\frac{1}{2C^{p+1}}$,代入式(11)可得

$$\phi'(t)\leqslant-\phi(t)+\left(2C^{p+1}\right)^2\phi(t)^p。\tag{12}$$

利用式(10),代入式(12)可得

$$\phi'(t)<-\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}+\left(2C^{p+1}\right)^2\phi(t)^p。\tag{13}$$

取 $\phi(t)=\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}$,代入式(13)右端式子中

$$-\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}+\left(2C^{p+1}\right)^2\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2p}{p-1}}=\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}\left[-1+\left(2C^{p+1}\right)^2\left(\frac{p+1}{2C^{p+1}}\right)^2\right]=\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}\left[-1+(p+1)^2\right]>0。$$

当 $\phi(t)$ 满足式(10)时,式(13)右端恒正。对式(13)移项,两边从0到 $t$ 积分可得

$$\int_{\phi(0)}^{\phi(t)}\frac{1}{-\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}}+\left(2C^{p+1}\right)^2\phi(t)^p}\mathrm{d}\phi<\int_0^t 1\mathrm{d}\tau,$$

令 $t\to T$ 有

$$T \geqslant \int_{||u_0||_{H_0^1}^2}^{\infty} \frac{1}{-\left(\frac{p+1}{2C^{p+1}}\right)^{\frac{2}{p-1}} + 4C^{2(p+1)}\phi(t)^p} \mathrm{d}\phi。$$

## 3 高初始能量的爆破时间下界估计

**定理8** 对于模型(M1),假设 $u_0 \in H_0^1(\Omega)$ 且满足 $J(u_0) < d, I(u_0) < 0$,则模型(M1)的解在有限时刻 $T$ 处爆破且有下界估计

$$T \geqslant \int_{||u_0||_{H_0^1}^2}^{\infty} \frac{1}{4C^{2(p+1)}\phi^p - C^{-\frac{2(p+1)}{p-1}}} \mathrm{d}\phi,$$

式中 $\phi(t) = ||u(\cdot, t)||_{H_0^1}^2$。

**证明** 定义泛函 $\Phi(t) = ||u(\cdot, t)||_{H_0^1}^2$,由 $J(u_0) > d, I(u_0) < 0$ 可得

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)||u_0||_{H_0^1}^2 > d。$$

根据引理5可知,$I(u) < 0$ 对任意的 $t \in [0, T)$ 总成立。因此

$$\frac{\mathrm{d}}{\mathrm{d}t}||u||_{H_0^1}^2 = -2I(u) > 0,$$

进而可得

$$||u||_{H_0^1}^2 > \frac{2(p+1)d}{p-1}。$$

由引理4有

$$\phi(t) > \left(\frac{1}{C}\right)^{\frac{2(p+1)}{p-1}}。 \tag{14}$$

与定理6证明类似,利用Young不等式,对任意的 $\epsilon > 0$,可得

$$\phi'(t) \leqslant (-2 + 2C^{p+1}\epsilon)\phi(t) + \frac{2C^{p+1}}{\epsilon}||u||_{H_0^1}^{2p}。$$

特别地,取 $\epsilon = \frac{1}{2C^{p+1}}$,同样有

$$\phi'(t) \leqslant -\phi(t) + (2C^{p+1})^2\phi(t)^p。 \tag{15}$$

将式(14)代入式(15)可得

$$\phi'(t) < -\left(\frac{1}{C}\right)^{\frac{2(p+1)}{p-1}} + (2C^{p+1})^2\phi(t)^p。 \tag{16}$$

取 $\phi(t) = \left(\frac{1}{C}\right)^{\frac{2(p+1)}{p-1}}$,代入式(16)右端,简单计算可得

$$-\left(\frac{1}{C}\right)^{\frac{2(p+1)}{p-1}} + (2C^{p+1})^2\left(\frac{1}{C}\right)^{\frac{2p(p+1)}{p-1}} = 3C^{\frac{-2(p+1)}{p-1}} > 0。$$

因此,当 $\phi(t)$ 满足式(14)时,不等式(16)右端恒正。

类似地,对式(16)经过移项,积分,取极限后,可得

$$T \geqslant \int_{||u_0||_{H_0^1}^2}^{\infty} \frac{1}{4C^{2(p+1)}\phi^p - C^{-\frac{2(p+1)}{p-1}}} \mathrm{d}\phi。$$

## 4 任意初始能量的爆破时间上界估计

**定理9** 对于模型(M1),假设 $u_0 \in H_0^1(\Omega)$ 且存在 $t_0 \in (0, T)$ 满足

$$J(u(t_0)) < \Lambda_{u(t_0)} := \frac{(p-1)}{2(p+1)}||u(t_0)||_{H_0^1}^2, \tag{17}$$

则模型(M1)的解在有限时刻 $T$ 处爆破,且 $T$ 有上界估计

$$T \leqslant \frac{8(p+1)||u_0||_{H_0^1}^2}{\left[(p-1)\|u_0\|_{H_0^1}^2 - 2(p+1)J(u_0)\right](p-1)^2}。 \tag{18}$$

**证明** 首先证明若 $u$ 是模型(M1)的解满足

$$J(u(t_0)) < \Lambda_{u(t_0)} := \frac{(p-1)}{2(p+1)}\left\|u(t_0)\right\|_{H_0^1}^2,$$

那么模型(M1)的解有限时间爆破。

$$\frac{1}{p+1}I(u(t_0)) = \frac{1}{p+1}||u(t_0)||_{H_0^1}^2 - \frac{1}{p+1}\left\|u(t_0)\right\|_{p+1}^{p+1} = J(u(t_0)) + \left(\frac{1}{p+1} - \frac{1}{2}\right)||u(t_0)||_{H_0^1}^2。$$

因此满足 $I(u_0) < 0$。根据引理1可知,模型(M1)的解有限时间爆破。下面给出爆破时间上界估计的证明。

定义泛函

$$M(t) = \int_{t_0}^{t} ||u(\tau)||_{H_0^1}^2 \mathrm{d}\tau + (T-t)\left\|u(t_0)\right\|_{H_0^1}^2 + \beta(t+\sigma)^2, \tag{19}$$

式中 $t \in [0, T]$, $\beta > 0$, $\sigma > 0$。对式(19)求导

$$M'(t) = \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}||u(\tau)||_{H_0^1}^2 \mathrm{d}\tau + 2\beta(t+\sigma) = 2\int_{t_0}^{t}(u, u_\tau) + (\nabla u, \nabla u_\tau)\mathrm{d}\tau + 2\beta(t+\sigma)。 \tag{20}$$

对式(20)求导

$$M''(t) = -2I(u(t)) + 2\beta = (p-1)||u||_{H_0^1}^2 - 2(p+1)J(u) + 2\beta > (p-1)||u||_{H_0^1}^2 - 2(p+1)J(u(t_0)) +$$

$$2(p+1)\int_{t_0}^{t}||u_\tau||_2^2\mathrm{d}\tau。$$

经过简单计算可得

$$M''(t)M(t)-(\theta+1)M'(t)^2=$$
$$M''(t)M(t)-$$
$$4(\theta+1)\left[\frac{1}{2}\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}||u(\tau)||_{H_0^1}^2\mathrm{d}\tau+\beta(t+\sigma)\right]^2,$$

式中

$$\left[\frac{1}{2}\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}||u(\tau)||_{H_0^1}^2\mathrm{d}\tau+\beta(t+\sigma)\right]^2=$$
$$\left[\int_{t_0}^{t}\int_{\Omega}u\cdot u_\tau+\nabla u\cdot\nabla u_\tau\mathrm{d}x\mathrm{d}\tau+\beta(t+\sigma)\right]^2\leqslant$$
$$\left[\int_{t_0}^{t}\int_{\Omega}\sqrt{|\nabla u|^2+|u|^2}\sqrt{|\nabla u_\tau|^2+|u_\tau|^2}\,\mathrm{d}x\mathrm{d}\tau+\beta(t+\sigma)\right]^2\leqslant$$
$$\left[\int_{t_0}^{t}||u||_{H_0^1}||u_\tau||_{H_0^1}\mathrm{d}\tau+\beta(t+\sigma)\right]^2\leqslant$$
$$\left[\left(\int_{t_0}^{t}||u||_{H_0^1}^2\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_{t_0}^{t}||u_\tau||_{H_0^1}^2\mathrm{d}\tau\right)^{\frac{1}{2}}+\beta(t+\sigma)\right]^2\leqslant$$
$$\left(\int_{t_0}^{t}||u_\tau||_{H_0^1}^2\mathrm{d}\tau+\beta\right)\cdot\left(\int_{t_0}^{t}||u||_{H_0^1}^2\mathrm{d}\tau+\beta(t+\sigma)^2\right)。\quad(21)$$

对式(21)简单计算可得

$$M''(t)M(t)-(\theta+1)M'(t)^2\geqslant$$
$$M(t)[2(p+1)(\frac{p-1}{2(p+1)}||u(t_0)||_{H_0^1}^2-$$
$$J(u(t_0))-4(\theta+1)\beta)+$$
$$(2(p+1)-4(\theta+1))\int_{t_0}^{t}||u_\tau||_{H_0^1}^2\mathrm{d}\tau]。\quad(22)$$

若$2(p+1)\geqslant 4(\theta+1)$,取$(\theta+1)=\frac{p+1}{2}$,即

$$M(t)M''(t)-\frac{p+1}{2}M'(t)^2\geqslant$$
$$2(p+1)M(t)[\frac{p-1}{2(p+1)}||u(t_0)||_{H_0^1}^2-$$
$$J(u(t_0))-2(p+1)\beta)]。$$

显然当

$$\beta\in\left[0,\frac{(p-1)}{2(p+1)}||u(t_0)||_{H_0^1}^2-J(u(t_0))\right]$$

时,满足

$$M''(t)M(t)-(\theta+1)M'(t)^2\geqslant 0。$$

根据引理4有

$$T\leqslant\frac{M(0)}{\theta M'(0)}=\frac{T||u(t_0)||_{H_0^1}^2+\beta\sigma^2}{(p-1)\beta\sigma},$$

即

$$T\left(1-\frac{||u(t_0)||_{H_0^1}^2}{(p-1)\beta\sigma}\right)<\frac{\sigma}{p-1}。\quad(23)$$

固定

$$\beta\in\left(0,\frac{(p-1)}{2(p+1)}||u(t_0)||_{H_0^1}^2-J(u(t_0))\right),$$

若从不等式(23)获得$T$的上界估计,需要满足

$$1-\frac{||u(t_0)||_{H_0^1}^2}{(p-1)\beta\sigma}>0,$$

即

$$\sigma\in\left(\frac{||u(t_0)||_{H_0^1}^2}{(p-1)\beta},+\infty\right)。\quad(24)$$

代入式(23)可得

$$T\leqslant\frac{\sigma}{p-1}\left(1-\frac{||u(t_0)||_{H_0^1}^2}{(p-1)\beta\sigma}\right)^{-1}=$$
$$\frac{\beta\sigma^2}{(p-1)\beta\sigma-||u(t_0)||_{H_0^1}^2}:=H(\beta,\sigma),$$

式中$H(\beta,\sigma)$的定义域满足

$$\Theta:=\left\{(\beta,\sigma)|\beta\in(0,\frac{p-1}{2(p+1)}||u(t_0)||_{H_0^1}^2-\right.$$
$$\left.J(u(t_0)),\ \sigma\in(\frac{||u(t_0)||_{H_0^1}^2}{(p-1)},+\infty)\right\}。$$

令

$$\beta=\frac{(p-1)}{2(p+1)}||u(t_0)||_{H_0^1}^2-J(u(t_0)),$$

由于$H(\beta,\sigma)$单调递减,则

$$T\leqslant\frac{A\sigma^2}{B\sigma-C}:=f(\sigma),$$

式中:

$$A=(p-1)||u(t_0)||_{H_0^1}^2-2(p+1)J(u(t_0)),$$
$$B=(p-1)^2||u(t_0)||_{H_0^1}^2-2(p^2-1)J(u(t_0)),$$
$$C=2(p+1)||u(t_0)||_{H_0^1}^2。$$

代入

$$\beta=\frac{(p-1)}{2(p+1)}\left\|u(t_0)\right\|_{H_0^1}^2-J(u(t_0)),$$

可得$\sigma\in\left(\frac{C}{B},\infty\right)$。对$f(\sigma)$求导,

$$f'(\sigma)=\frac{AB\sigma^2-2AC\sigma}{(B\sigma-C)^2}。$$

若$f'(\sigma)=0$,解得$\sigma=0,\sigma=\frac{2C}{B}$。其中

$$f\left(\frac{2C}{B}\right)=\frac{8(p+1)\|u(t_0)\|_{H_0^1}^2}{\left[(p-1)\|u(t_0)\|_{H_0^1}^2-2(p+1)J(u(t_0))\right](p-1)^2}。$$

因此

$$T\leqslant\frac{8(p+1)\left\|u(t_0)\right\|_{H_0^1}^2}{\left[(p-1)\left\|u(t_0)\right\|_{H_0^1}^2-2(p+1)J(u(t_0))\right](p-1)^2}。$$

## 5 结束语

本文研究了一类半线性伪抛物方程解的爆破行为。基于现有的爆破准则,分别考虑了模型在低初始能量,高初始能量下解的爆破时间下界估计以及模型在任意初始能量下解的爆破时间上界估计。通过构造合适的辅助函数,获得相关微分不等式,从而获得下界估计。利用凸方法,获得上界估计。对于初值满足其他条件时,该模型爆破解的时间估计有待进一步研究。

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