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大尺度湿大气动力学方程组整体强解的存在唯一性*

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摘要：研究了具有地形因素的大尺度湿大气动力学方程组的初边值问题，在假设初值满足二阶正则性时，利用能量估计和迭代方法，证明了初边值问题的二阶正则整体强解的存在唯一性。

关键词：湿大气方程组；二阶正则；强解；地形
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Global existence and uniqueness of strong solution to the large-scale moist atmospheric dynamic equations*

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Abstract：This paper studied the initial-boundary value problems to the large-scale moist atmospheric dynamic equations with the effects of topography. Assuming the initial data satisfies second-order regularity，by the energy method and iterative method，the global existence and uniqueness of strong solution with second-order regular can be proved.

Keywords：moist atmospheric dynamic equations；second-order regular；strong solution；topography
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0　引　言

很多数学家已经研究了原始方程组的数学理论。Lions等[1-3]在1992—1995年证明了海气耦合方程组弱解的整体存在性等；Guillén-González等[4]在2001年、Temam和Ziane[5]在2005年、Cao和Titi[6]在2007年分别证明了原始方程组初边值问题的一阶局部和整体强解的存在唯一性；Guo和Huang[7]在2009年证明了大气方程组初边值问题弱强解，二阶强解的整体存在唯一性；Cao等[8-12]在2014—2020年证明了具有部分耗散的原始方程组初边值问题一阶和二阶正则强解的整体存在唯一性。此外，还有一些有关原始方程组初边值问题的研究成果，参见文献[13-16]。

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在上述所研究的大气和海洋原始方程组中都采取了很多与实际物理过程不相符合的近似。如:取大气上界气压为某一正常数;直接给出狄利克雷边界条件或诺依曼边界条件;未消除整层无辐散近似;取外源强迫项为已知函数,所以早在1998年曾庆存[17]就针对这些近似进行了改进;2001年Wu等[18]、2007年黄海洋和郭柏灵[19]、2018年Lian和Zeng[20]证明了改进的大气方程组初边值问题整体弱解的适定性以及吸引子的整体存在性;2018年Lian等[21]、2020年Lian和Ma[22]分别得到了改进的干大气和湿大气方程组初边值问题一阶正则强解的整体存在唯一性;2022年Ma等[23]又证明了改进的湿大气方程组初边值问题二阶正则强解的局部存在唯一性。另外,还有一些有关大气,海洋动力学方程的数学理论研究成果参见文献[24-32]。

1　预备知识

本文研究了文献[17,28]中加入地形因素的湿大气方程组的初边值问题。此方程组是在地形坐标系(θ,λ,ζ,t)下给出的,其中$\theta\in[0,\pi]$是余纬,$\lambda\in[0,2\pi]$是经度,ζ是地形变量,$\zeta=(p-p_t)/(p_s-p_t)$,$p\in[p_t,p_s]$是大气压力,$p_s(\theta,\lambda,t)$是地球表面的大气压力,常数$p_t>0$是顶层大气的压力,t是时间。湿大气动力学方程组的状态变量分别为大气水平速度$(V)(V=(V_\theta,V_\lambda))$、温度偏差量$(T')$、比湿$(q)$、液态水含量$(m_w)$,下面给出湿大气动力学方程组以及初边值条件。

水平速度方程为

$$\frac{\partial V}{\partial t}+\dot{\zeta}^*\frac{\partial V}{\partial\zeta}+\left(2\omega\cos\theta+\frac{\cot\theta}{a}v_\lambda\right)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V+(V^*\cdot\nabla)V+R\nabla\int_\zeta^1\frac{\widetilde{p_{es}}T'(s)}{\widetilde{p_{es}}s+p_t}\mathrm{d}s+\frac{R\zeta}{\widetilde{p_{es}}\zeta+p_t}T'(s)\nabla\widetilde{p_{es}}-\nabla\left(\frac{R\widetilde{T}_s}{\widetilde{p_{es}}}\int_0^t\nabla\cdot\left(\widetilde{p_{es}}\overline{V}\right)\mathrm{d}\tau\right)=\frac{\mu_1}{\widetilde{p_{es}}}\Delta V+v_1\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial V}{\partial\zeta}\right),\tag{1.1}$$

温度偏差量方程为

$$\frac{R}{c_0^2}\left(\frac{\partial T'}{\partial t}+(V^*\cdot\nabla)T'+\dot{\zeta}^*\frac{\partial T'}{\partial\zeta}\right)+\frac{R}{\widetilde{p_{es}}\zeta+p_t}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{es}}V\right)\mathrm{d}s-\frac{R\zeta}{\widetilde{p_{es}}\zeta+p_t}\nabla\widetilde{p_{es}}\cdot V=\frac{R\mu_2}{\widetilde{p_{es}}c_0^2c_P}\Delta T'+\frac{Rv_2}{c_0^2c_P}\frac{\partial}{\partial\zeta}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial T'}{\partial\zeta}-\frac{R}{c_0^2}\kappa_aT'-\frac{RL^2}{c_0^2}\delta_{21}\delta_{22}\left(W(T)\left(\frac{\dot{\zeta}}{\widetilde{p_{es}}\zeta+p_t}\right)\right),\tag{1.2}$$

比湿方程为

$$\frac{\partial q}{\partial t}+(V^*\cdot\nabla)q-\left(\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{es}}V^*\right)\mathrm{d}s\right)\frac{\partial q}{\partial\zeta}=\frac{\mu_3}{\widetilde{p_{es}}}\Delta q+v_3\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial q}{\partial\zeta}\right)+\delta_{21}\delta_{22}\left(LW(T)\frac{\widetilde{p_{es}}\dot{\zeta}}{\widetilde{p_{es}}\zeta+p_t}\right),\tag{1.3}$$

含水量方程为

$$\frac{\partial m_w}{\partial t}+(V^*\cdot\nabla)m_w-\left(\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{es}}V^*\right)\mathrm{d}s\right)\frac{\partial m_w}{\partial\zeta}=v_4\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial m_w}{\partial\zeta}\right)+\frac{\mu_4}{\widetilde{p_{es}}}\Delta m_w-\delta_{21}\delta_{22}\left(LW(T)\frac{\widetilde{p_{es}}\dot{\zeta}}{\widetilde{p_{es}}\zeta+p_t}\right)+h_1\left(\delta_{21}\delta_{22}\left(LW(T)\frac{\widetilde{p_{es}}\dot{\zeta}}{\widetilde{p_{es}}\zeta+p_t}\right)\right),\tag{1.4}$$

初边值条件为

$$\begin{cases}U|_{t=0}=(v_\theta,v_\lambda,T',q,m_w)\big|_{t=0}=(v_{\theta_0},v_{\lambda_0},T_0',q_0,m_{w_0})=0,\\U(\theta,\lambda,\zeta)=U(\theta+\pi,\lambda,\zeta)=U(\theta,\lambda+2\pi,\zeta),\ U_\zeta\big|_{\zeta=0,1}=0,\end{cases}\tag{1.5}$$

式中:$\widetilde{T}(\zeta)$是参考标准温度;$\widetilde{p_{es}}(\theta,\lambda)$是参考标准地表压力;$\widetilde{T_s}(\theta,\lambda)$是地球的参考标准表面温度;$\omega$是地球自转角速度;$g$是重力加速度;$L$是相变潜热;$\kappa_a$是正常数;$\delta_{21}$和$\delta_{22}$分别是0或1;$c_0$,$c_p$,$R$和$R_v$是热力学参数;$\mu_i$,$v_i(i=1,2,3,4)$为耗散系数。$W(T)$是一个全局的Lipschitz有界函数,表达式为

$$W(T)=q_mT\left(\frac{RL-c_pR_vT}{c_pR_vT^2-L^2q_m}\right),\tag{1.6}$$

式中q_m是一个正的常数。

方程组中的$h_1(x)$是线性函数,表达式为

$$h_1(x)=\begin{cases}\alpha x-\beta, & x<0,\\0, & x>0,\end{cases}\tag{1.7}$$

式中:$0<\alpha<1$;$\beta>0$,可参见文献[17-18,20,22-23,28]。

另外,采用文献[18,22]中的方程式可以给出修正的光滑速度场$(V^*,\dot{\zeta}^*)$的具体形式。

令$\bar{V}\coloneqq\int_0^1 V(\theta,\lambda,\zeta,t)\mathrm{d}\zeta$,然后将$\widetilde{p_{es}}\bar{V}$分解为无旋度场和无散度场,$\widetilde{p_{es}}\bar{V}=\nabla\chi+\nabla\psi^{\perp}$,则有$V^*=V-\widetilde{p_{es}}^{-1}\nabla\chi=(v_\theta^*,v_\lambda^*)$,$\dot{\zeta}^*=-\widetilde{p_{es}}^{-1}\int_0^\zeta\nabla\cdot(\widetilde{p_{es}}V^*)\mathrm{d}s$,且当$\zeta=1$时,$\dot{\zeta}^*=0$。

2　主要结论

首先给出二阶正则整体强解的定义。

定义1　若$U_0=(V_0,T_0',q_0,m_{w0})\in H^2(\Omega)$,对于任意给定的$T>0$,如果$U$在弱的意义下满足初边值问题式(1.1~1.5),且满足

$$U=(V,T',q,m_w)\in L^\infty([0,T];\ H^2(\Omega))\cap L^2([0,T];\ H^3(\Omega)),$$
$$(V_t,T_t',q_t,m_{wt})\in L^2([0,T];\ H^1(\Omega)),$$

则U称为初边值问题式(1.1~1.5)在$\Omega\times[0,T]$上的强解。

下面再给出初边值问题式(1.1~1.5)二阶正则整体强解的存在唯一性结论:

定理1　若$U_0\in H^2(\Omega)$,且满足以下条件:

$$\tilde{T}(\zeta),\tilde{T}^{-1}(\zeta)\in W^{2,\infty}([0,1]),\tilde{T}(\zeta)=O(\zeta)\geqslant 0,$$
$$\widetilde{p_{es}}(\theta,\lambda),\widetilde{p_{es}}^{-1}(\theta,\lambda),\widetilde{T_s}(\theta,\lambda)\in W^{3,\infty}(S^2),$$

则对任意给定的$T>0$,初边值问题式(1.1~1.5)在$\Omega\times[0,T]$上存在唯一的整体强解U。

3　结论证明

本文采用文献[7,29]中的迭代方法证明了二阶正则整体强解的存在性,具体思路如下:首先,利用文献[22-23]中的方法可得到变量V_0、T_0'、q_0、m_{w0}的H^1估计和湿大气方程组局部强解的存在唯一性;其次,利用能量估计方法可证明$V_\zeta,T_\zeta,q_\zeta,m_{w\zeta}\in L^\infty([0,T];\ L^4(\Omega))$;再次,在此基础之上证明了$V_t,T_t,q_t,m_{wt}\in L^\infty([0,T];\ L^2(\Omega))$;最后,利用迭代方法证明$V,T,q,m_w\in L^\infty([0,T];\ H^2(\Omega))$。下面给出相关引理。

引理1　由文献[22]可得初边值问题式(1.1~1.5)的一阶正则整体强解存在唯一且对于任意给定的$T>0$,满足

$$U\in L^\infty([0,T];\ H^1(\Omega))\cap L^2([0,T];\ H^2(\Omega))。\tag{3.1}$$

引理2　在定理1的条件下,对于任意给定的$T>0$,V_ζ满足

$$V_\zeta\in L^\infty([0,T];\ L^4(\Omega))。\tag{3.2}$$

证明　将式(1.1)关于ζ求导,有

$$\frac{\partial V_\zeta}{\partial t}+(V^*\cdot\nabla)V_\zeta+(V^*{}_\zeta\cdot\nabla)V-\left(\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot(\widetilde{p_{es}}V^*)\mathrm{d}s\right)V_{\zeta\zeta}+$$
$$(2\omega\cos\theta)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V_\zeta+R\frac{\partial}{\partial\zeta}\left(\nabla\int_\zeta^1\frac{\widetilde{p_{es}}T'(s)}{\widetilde{p_{es}}s+p_t}\mathrm{d}s\right)+$$
$$\frac{\partial}{\partial\zeta}\left(\frac{R\zeta}{\widetilde{p_{es}}\zeta+p_t}T'\right)\nabla\widetilde{p_{es}}-\frac{1}{\widetilde{p_{es}}}\nabla\cdot(\widetilde{p_{es}}V^*)V_\zeta$$
$$\frac{\partial}{\partial\zeta}\left(\frac{\cot\theta}{a}v_\lambda\begin{pmatrix}0&-1\\1&0\end{pmatrix}V\right)=$$
$$\frac{\mu_1}{\widetilde{p_{es}}}\Delta V_\zeta+\nu_1\frac{\partial}{\partial\zeta}\left(\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial V}{\partial\zeta}\right)\right),\tag{3.3}$$

将式(3.3)与$\widetilde{p_{es}}|V_\zeta|^2V_\zeta$做内积,可得

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{p_{es}}|V_\zeta|^4\mathrm{d}\sigma\mathrm{d}\zeta+3\nu_1\int_\Omega\widetilde{p_{es}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left(|V_{\zeta\zeta}|\right)^2|V_\zeta|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$
$$3\mu_1\int_\Omega|\nabla V_\zeta|^2|V_\zeta|^2\mathrm{d}\sigma\mathrm{d}\zeta=I_1+I_2+I_3+I_4+I_5+I_6,\tag{3.4}$$

经计算有

$$I_1\coloneqq-\int_\Omega\left((V^*\cdot\nabla)V_\zeta-\left(\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot(\widetilde{p_{es}}V^*)\mathrm{d}s\right)V_{\zeta\zeta}\right)\cdot\left(\widetilde{p_{es}}|V_\zeta|^2V_\zeta\right)\mathrm{d}\sigma\mathrm{d}\zeta=0,\tag{3.5}$$

利用Gagliardo-Nirenberg-Sobolev不等式,Young不等式,$V_\zeta^*=V_\zeta$,,可得

$$I_2\coloneqq$$
$$-\int_\Omega\left((V_\zeta^*\cdot\nabla)V-\frac{1}{\widetilde{p_{es}}}\nabla\cdot(\widetilde{p_{es}}V^*)V_\zeta\right)\cdot\left(\widetilde{p_{es}}|V_\zeta|^2V_\zeta\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$
$$C\int_\Omega\left(|V|\|\nabla V_\zeta\||V_\zeta|^3+|V^*|\|\nabla V_\zeta\||V_\zeta|^3\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$
$$C\left(\|V\|_{L^4(\Omega)}+\|V^*\|_{L^4(\Omega)}\right)\||V_\zeta|^2\|_{L^2(\Omega)}^{\frac{1}{4}}\cdot$$
$$\left(\||V_\zeta|^2\|_{H^1(\Omega)}\right)^{\frac{3}{4}}\|\nabla V_\zeta\||V_\zeta|\|_{L^2(\Omega)}\leqslant$$

$$C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+\varepsilon\left\|\left|V_\zeta\right|^2\right\|_{H^1(\Omega)}^2, \tag{3.6}$$

此处C是一个与初值无关的正常数,ε是一个足够小的正常数,下同,

$$I_3=-\int_\Omega\left((2\omega\cos\theta)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V_\zeta+\frac{\partial}{\partial\zeta}\left(\frac{\cot\theta}{a}v_\lambda\begin{pmatrix}0&-1\\1&0\end{pmatrix}V\right)\right)\cdot\left(\widetilde{p_{\mathrm{es}}}\left|V_\zeta\right|^2V_\zeta\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+C\left\|V\right\|_{L^4(\Omega)}\left\|\left|V_\zeta\right|^2\right\|_{L^2(\Omega)}^{\frac14}\cdot\left(\left\|\left|V_\zeta\right|^2\right\|_{H^1(\Omega)}\right)^{\frac34}\left\|\left|V_\zeta\right|^2\right\|_{L^2(\Omega)}\leqslant C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+\varepsilon\left\|\left|V_\zeta\right|^2\right\|_{H^1(\Omega)}^2, \tag{3.7}$$

$$I_4=-\int_\Omega\left(R\frac{\partial}{\partial\zeta}\left(\nabla\int_\zeta^1\frac{\widetilde{p_{\mathrm{es}}}T'(\mathrm{s})}{\widetilde{p_{\mathrm{es}}}s+p_t}\mathrm{d}s\right)\right)\cdot\left(\widetilde{p_{\mathrm{es}}}\left|V_\zeta\right|^2V_\zeta\right)\mathrm{d}\sigma\mathrm{d}\zeta=\int_\Omega R\nabla\left(\frac{\widetilde{p_{\mathrm{es}}}T'}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\right)\cdot\left(\widetilde{p_{\mathrm{es}}}\left|V_\zeta\right|^2V_\zeta\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\int_\Omega R\left|\left(\frac{\widetilde{p_{\mathrm{es}}}T'}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\right)\right|\left(\left|\nabla\widetilde{p_{\mathrm{es}}}\right|\left|V_\zeta\right|^3+\widetilde{p_{\mathrm{es}}}\left|\nabla V_\zeta\right|\left|V_\zeta\right|^2\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\left\|T'\right\|_{L^4(\Omega)}^4+C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+\varepsilon\left(\left\|V_\zeta\cdot\nabla V_\zeta\right\|_{L^2(\Omega)}^2\right), \tag{3.8}$$

$$I_5=-\int_\Omega\left(\frac{\partial}{\partial\zeta}\left(\frac{R\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}T'\right)\nabla\widetilde{p_{\mathrm{es}}}\right)\cdot\left(\widetilde{p_{\mathrm{es}}}\left|V_\zeta\right|^2V_\zeta\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\left(\left\|T_\zeta'\right\|_{L^2(\Omega)}+\left\|T'\right\|_{L^2(\Omega)}\right)\left\|\left|V_\zeta\right|^2\right\|_{L^2(\Omega)}^{\frac14}\left(\left\|\left|V_\zeta\right|^2\right\|_{H^1(\Omega)}\right)^{\frac34}\left\|V_\zeta\right\|_{L^4(\Omega)}\leqslant C+C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+\varepsilon\left\|\left|V_\zeta\right|^2\right\|_{H^1(\Omega)}^2, \tag{3.9}$$

$$I_6=-\int_\Omega\left(3\nu_1\widetilde{p_{\mathrm{es}}}\frac{\partial}{\partial\zeta}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left|V_{\zeta\zeta}\right|\left|V_\zeta\right|^3\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+\varepsilon\left(\left\|\left|V_\zeta\right|\left|V_{\zeta\zeta}\right|\right\|_{L^2(\Omega)}^2\right), \tag{3.10}$$

选取充分小的ε,由式(3.4~3.10)得

$$\frac14\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{p_{\mathrm{es}}}\left|V_\zeta\right|^4\mathrm{d}\sigma\mathrm{d}\zeta+3\mu_1\int_\Omega\left|\nabla V_\zeta\right|^2\left|V_\zeta\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+2\nu_1\int_\Omega\widetilde{p_{\mathrm{es}}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left(\left|V_{\zeta\zeta}\right|\right)^2\left|V_\zeta\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C+C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+C\left\|T'\right\|_{L^4(\Omega)}^4, \tag{3.11}$$

利用Gronwall不等式和式(3.1),可得$V_\zeta\in L^\infty\left([0,T];\ L^4(\Omega)\right)$。

引理3 在定理1的条件下,对于任意给定的$T>0$,T_ζ',q_ζ,$m_{w\zeta}$满足

$$T_\zeta',\ q_\zeta,\ m_{w\zeta}\in L^\infty\left([0,T];\ L^4(\Omega)\right)。 \tag{3.12}$$

证明 将式(1.2)关于ζ求导,有

$$\frac{R}{c_0^2}\left(\frac{\partial T_\zeta'}{\partial t}+\left(V_\zeta^*\cdot\nabla\right)T'-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\right)\frac{\partial T'}{\partial\zeta}\right)+\frac{R}{c_0^2}\left(\left(V^*\cdot\nabla\right)T_\zeta'-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right)T_{\zeta\zeta}'\right)+\frac{\partial}{\partial\zeta}\left(\frac{R}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V\right)\mathrm{d}s\right)-\frac{\partial}{\partial\zeta}\left(\frac{R\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\nabla\widetilde{p_{\mathrm{es}}}\cdot V\right)=\frac{R\mu_2}{\widetilde{p_{\mathrm{es}}}c_0^2c_P}\Delta T_\zeta'+\frac{R\nu_2}{c_0^2c_P}\frac{\partial}{\partial\zeta}\left(\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial T'}{\partial\zeta}\right)\right)+\frac{R}{c_0^2}\frac{\partial}{\partial\zeta}\left(-\frac{R}{c_0^2}\kappa_aT'-\frac{RL^2}{c_0^2}\delta_{21}\delta_{22}\left(W(T)\left(\frac{\dot\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\right)\right)\right), \tag{3.13}$$

将式(3.13)与$\widetilde{p_{\mathrm{es}}}T_\zeta'\left|T_\zeta'\right|^2$做$L^2$内积,有

$$\frac{R}{4c_0^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{p_{\mathrm{es}}}\left|T_\zeta'\right|^4\mathrm{d}\sigma\mathrm{d}\zeta+\frac{3R\mu_2}{c_0^2c_P}\int_\Omega\left|\nabla T_\zeta'\right|^2\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+\frac{3R\nu_2}{c_0^2c_P}\int_\Omega\widetilde{p_{\mathrm{es}}}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\right)\left|T_{\zeta\zeta}'\right|^2\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta=J_1+J_2+J_3+J_4+J_5+J_6, \tag{3.14}$$

经计算可得

$$J_1=-\frac{R}{c_0^2}\int_\Omega\left(\left(V^*\cdot\nabla\right)T_\zeta'-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right)T_{\zeta\zeta}'\right)\cdot\widetilde{p_{\mathrm{es}}}T_\zeta'\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta=0, \tag{3.15}$$

类似于式(3.5)的证明可得

$$J_2=-\frac{R}{c_0^2}\int_\Omega\left(\left(V_\zeta^*\cdot\nabla\right)T'-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\right)\frac{\partial T'}{\partial\zeta}\right)\widetilde{p_{\mathrm{es}}}T_\zeta'\left|T_\zeta'\right|^2\cdot\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\left\|T_\zeta'\right\|_{L^4(\Omega)}^4+C\left\|V_\zeta\right\|_{L^4(\Omega)}^4+C\left\|\left|V_\zeta\right|^2\right\|_{H^1(\Omega)}^2+\varepsilon\left\|\left|T_\zeta'\right|^2\right\|_{H^1(\Omega)}^2, \tag{3.16}$$

还可得

$$J_3=-\int_\Omega\left(\frac{\partial}{\partial\zeta}\left(\frac{R}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V\right)\mathrm{d}s\right)\right)\widetilde{p_{\mathrm{es}}}T_\zeta'\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta=-\int_\Omega\left(\frac{\partial}{\partial\zeta}\left(\frac{R}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\right)\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V\right)\mathrm{d}s+\right.$$

$$\frac{R}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V\right)\Bigg)\cdot\left(\widetilde{p_{\mathrm{es}}}T_\zeta'\left|T_\zeta'\right|^2\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left(\|V\|_{L^2(\Omega)}+\|\nabla V\|_{L^2(\Omega)}\right)\left\|T_\zeta'\right\|_{L^4(\Omega)}\left\|\left|T_\zeta'\right|^2\right\|_{L^2(\Omega)}^{\frac{1}{4}}\left\|\left|T_\zeta'\right|^2\right\|_{H^1(\Omega)}^{\frac{3}{4}}\leqslant$$

$$C+C\left\|T_\zeta'\right\|_{L^4(\Omega)}^4+\varepsilon\left\|\left|T_\zeta'\right|^2\right\|_{H^1(\Omega)}^2,\qquad(3.17)$$

$$J_4:=\int_\Omega\left(\frac{\partial}{\partial\zeta}\left(\frac{R\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\nabla\widetilde{p_{\mathrm{es}}}\cdot V\right)\right)\widetilde{p_{\mathrm{es}}}T_\zeta'\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left(\|V_\zeta\|_{L^2(\Omega)}+\|V\|_{L^2(\Omega)}\right)\left\|T_\zeta'\right\|_{L^4(\Omega)}\left\|\left|T_\zeta'\right|^2\right\|_{L^2(\Omega)}^{\frac{1}{4}}\left(\left\|\left|T_\zeta'\right|^2\right\|_{H^1(\Omega)}\right)^{\frac{3}{4}}\leqslant$$

$$C+C\left\|T_\zeta'\right\|_{L^4(\Omega)}^4+\varepsilon\left\|\left|T_\zeta'\right|^2\right\|_{H^1(\Omega)}^2,\qquad(3.18)$$

$$J_5:=\int_\Omega\frac{\partial}{\partial\zeta}\frac{R}{c_0^2}\left(-\frac{R}{c_0^2}\kappa_aT'-\frac{RL^2}{c_0^2}\delta_{21}\delta_{22}\left(W(T)\left(\frac{\dot\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}\right)\right)\right)\cdot$$

$$\widetilde{p_{\mathrm{es}}}T_\zeta'\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\left\|T_\zeta'\right\|_{L^4(\Omega)}^4+$$

$$C\|\nabla V\|_{L^2(\Omega)}^{\frac{1}{2}}\left(\|\Delta V\|_{L^2(\Omega)}^{\frac{1}{2}}+\|\nabla V\|_{L^2(\Omega)}^{\frac{1}{2}}\right)\left\|T_\zeta'\right\|_{L^2(\Omega)}^{\frac{1}{2}}$$

$$\left(\left\|T_\zeta'\right\|_{L^2(\Omega)}^{\frac{1}{2}}+\left\|\nabla T_\zeta'\right\|_{L^2(\Omega)}^{\frac{1}{2}}\right)\left\|\left|T_{\zeta\zeta}'\right|\left|T_\zeta'\right|\right\|_{L^2(\Omega)}\leqslant$$

$$C+C\left\|T_\zeta'\right\|_{L^4(\Omega)}^4+C\|\Delta V\|_{L^2(\Omega)}^2+C\left\|\nabla T_\zeta'\right\|_{L^2(\Omega)}^2\left\|T_\zeta'\right\|_{L^2(\Omega)}^2+$$

$$\varepsilon\left\|\left|T_{\zeta\zeta}'\right|\left|T_\zeta'\right|\right\|_{L^2(\Omega)}^2,\qquad(3.19)$$

$$J_6:=-\int_\Omega\frac{3Rv_2\widetilde{p_{\mathrm{es}}}}{c_0^2c_P}\left(\frac{\partial}{\partial\zeta}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left|T_{\zeta\zeta}'\right|\left|T_\zeta'\right|^3\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|T_\zeta'\right\|_{L^4(\Omega)}^4+\varepsilon\left\|\left|T_{\zeta\zeta}'\right|\left|T_\zeta'\right|\right\|_{L^2(\Omega)}^2,\qquad(3.20)$$

选取充分小的ε,由式(3.14~3.20)得

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{p_{\mathrm{es}}}\left|T_\zeta'\right|^4\mathrm{d}\sigma\mathrm{d}\zeta+\int_\Omega\left|\nabla T_\zeta'\right|^2\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$\int_\Omega\widetilde{p_{\mathrm{es}}}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\right)\left|T_{\zeta\zeta}'\right|^2\left|T_\zeta'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C+C\left\|T_\zeta'\right\|_{L^4(\Omega)}^4+C\left\|\left|V_\zeta\right|^2\right\|_{H^1(\Omega)}^2,\qquad(3.21)$$

利用Gronwall不等式,可得$T_\zeta\in L^\infty\left([0,T];\ L^4(\Omega)\right)$,利用类似的方法,还可以证得

$$q_\zeta,m_{w\zeta}\in L^\infty\left([0,T];\ L^4(\Omega)\right)。$$

引理4　在定理1的条件下,对于任意给定的$T>0$,有如下估计成立

$$V_t,T_t',q_t,m_{wt}\in L^\infty\left([0,T];\ L^2(\Omega)\right)。\qquad(3.22)$$

证明　将式(1.1)关于t求导,有

$$\frac{\partial V_t}{\partial t}+\left(V^*\cdot\nabla\right)V_t+\left(V_t^*\cdot\nabla\right)V-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right)V_{\zeta t}-$$

$$\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V_t^*\right)\mathrm{d}sV_\zeta+(2\omega\cos\theta)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V_t+$$

$$\frac{\cot\theta}{a}v_{\lambda t}\begin{pmatrix}0&-1\\1&0\end{pmatrix}V+\frac{\cot\theta}{a}v_\lambda\begin{pmatrix}0&-1\\1&0\end{pmatrix}V_t+$$

$$R\left(\nabla\int_\zeta^1\frac{\widetilde{p_{\mathrm{es}}}T_t'(s)}{\widetilde{p_{\mathrm{es}}}s+p_t}\mathrm{d}s\right)+\left(\frac{R\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}T_t'\right)\nabla\widetilde{p_{\mathrm{es}}}-$$

$$\frac{\partial}{\partial t}\left(\nabla\left(\frac{R\widetilde{T_s}}{\widetilde{p_{\mathrm{es}}}}\int_0^t\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}\overline{V}\right)\mathrm{d}\tau\right)\right)=$$

$$\frac{\mu_1}{\widetilde{p_{\mathrm{es}}}}\Delta V_t+\nu_1\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial V_t}{\partial\zeta}\right),\qquad(3.23)$$

将式(3.23)与$\widetilde{p_{\mathrm{es}}}V_t$做内积,得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{p_{\mathrm{es}}}\left|V_t\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+\nu_1\int_\Omega\widetilde{p_{\mathrm{es}}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left|V_{t\zeta}\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$\mu_1\int_\Omega\left|\nabla V_t\right|^2\mathrm{d}\sigma\mathrm{d}\zeta=$$

$$K_1+K_2+K_3+K_4+K_5+K_6,\qquad(3.24)$$

经计算可得

$$K_1:=-\int_\Omega\left(\left(V^*\cdot\nabla\right)V_t-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right)V_{\zeta t}\right)\cdot$$

$$\left(\widetilde{p_{\mathrm{es}}}V_t\right)\mathrm{d}\sigma\mathrm{d}\zeta=0,\qquad(3.25)$$

利用式(3.1)以及Hölder不等式和Young不等式,可得如下估计:

$$K_2:=-\int_\Omega\left(\left(V_t^*\cdot\nabla\right)V-\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^\zeta\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V_t^*\right)\mathrm{d}sV_\zeta\right)\cdot\left(\widetilde{p_{\mathrm{es}}}V_t\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\int_\Omega\left(\left|V_t^*\right|\left|\nabla V\right|\left|V_t\right|+\int_0^1\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V_t^*\right)\mathrm{d}s\left|V_\zeta\right|\left|V_t\right|\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|V_t\right\|_{L^2(\Omega)}^{\frac{1}{2}}\left\|V_t\right\|_{H^1(\Omega)}^{\frac{3}{2}}\left\|\nabla V\right\|_{L^2(\Omega)}+$$

$$C\left\|V_t\right\|_{L^2(\Omega)}^{\frac{1}{4}}\left\|V_t\right\|_{H^1(\Omega)}^{\frac{3}{4}}\left\|V_\zeta\right\|_{L^4(\Omega)}\left\|\nabla V_t\right\|_{L^2(\Omega)}\leqslant$$

$$C\left\|V_t\right\|_{L^2(\Omega)}^2+\varepsilon\left(\left\|V_t\right\|_{H^1(\Omega)}^2\right),\qquad(3.26)$$

$$K_3:=-\int_\Omega\left((2\omega\cos\theta)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V_t+\frac{\cot\theta}{a}v_{t\lambda}\begin{pmatrix}0&-1\\1&0\end{pmatrix}V+\right.$$

$$\frac{\cot\theta}{a}v_\lambda\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}V_t\Big)\cdot\Big(\widetilde{p_{es}}V_t\Big)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|V_t\right\|_{L^2(\Omega)}^{\frac{1}{2}}\left\|V_t\right\|_{H^1(\Omega)}^{\frac{3}{2}}\left\|V\right\|_{L^2(\Omega)}\leqslant$$

$$C\left\|V_t\right\|_{L^2(\Omega)}^2+\varepsilon\left(\left\|V_t\right\|_{H^1(\Omega)}^2\right),\tag{3.27}$$

$$K_4:=-\int_\Omega\left(R\frac{\partial}{\partial t}\left(\nabla\int_\zeta^1\frac{\widetilde{p_{es}}T'(s)}{\widetilde{p_{es}}s+p_t}\mathrm{d}s\right)\right)\cdot\Big(\widetilde{p_{es}}V_t\Big)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$\int_\Omega\left|R\int_0^1\frac{\widetilde{p_{es}}T_t'(s)}{\widetilde{p_{es}}s+p_t}\mathrm{d}s\right|\left(\left|\nabla\widetilde{p_{es}}\right|\left|V_t\right|+\widetilde{p_{es}}\left|\nabla V_t\right|\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|T_t'\right\|_{L^2(\Omega)}^2+C\left\|V_t\right\|_{L^2(\Omega)}^2+\varepsilon\left\|\nabla V_t\right\|_{L^2(\Omega)}^2,\tag{3.28}$$

$$K_5:=-\int_\Omega\left(\frac{\partial}{\partial t}\left(\frac{R\zeta}{\widetilde{p_{es}}\zeta+p_t}T'\right)\nabla\widetilde{p_{es}}\right)\cdot\Big(\widetilde{p_{es}}V_t\Big)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|T_t'\right\|_{L^2(\Omega)}^2+C\left\|V_t\right\|_{L^2(\Omega)}^2,\tag{3.29}$$

$$K_6:=\int_\Omega\left(\frac{\partial}{\partial t}\left(\nabla\left(\frac{R\widetilde{T}_s}{\widetilde{p_{es}}}\int_0^t\nabla\cdot\Big(\widetilde{p_{es}}\overline{V}\Big)\mathrm{d}\tau\right)\right)\right)\cdot\Big(\widetilde{p_{es}}V_t\Big)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\int_\Omega\left|\frac{R\widetilde{T}_s}{\widetilde{p_{es}}}\nabla\cdot\Big(\widetilde{p_{es}}\overline{V}\Big)\right|\left(\left|\nabla\widetilde{p_{es}}\right|\left|V_t\right|+\widetilde{p_{es}}\left|\nabla V_t\right|\right)\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C+C\int_\Omega\left(\left|\overline{V}\right|^2+\left|\nabla\overline{V}\right|^2\right)\mathrm{d}\sigma\mathrm{d}\zeta+C\left\|V_t\right\|_{L^2(\Omega)}^2+\varepsilon\left\|\nabla V_t\right\|_{L^2(\Omega)}^2,\tag{3.30}$$

选取充分小的ε,由式(3.24~3.30)得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{p_{es}}\left|V_t\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+\mu_1\int_\Omega\left|\nabla V_t\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$v_1\int_\Omega\widetilde{p_{es}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2k_{s1}\left|V_{t\zeta}\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C+C\left\|V_t\right\|_{L^2(\Omega)}^2+C\left\|T_t'\right\|_{L^2(\Omega)}^2,\tag{3.31}$$

将式(1.2)关于t求导,有

$$\frac{R}{c_0^2}\left(\frac{\partial T_t'}{\partial t}+\Big(V_t^*\cdot\nabla\Big)T'+\Big(V^*\cdot\nabla\Big)T_t'-\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot\Big(\widetilde{p_{es}}V_t^*\Big)\mathrm{d}s\frac{\partial T'}{\partial\zeta}\right)+$$

$$\frac{R}{c_0^2}\left(\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot\Big(\widetilde{p_{es}}V^*\Big)\mathrm{d}s\right)\frac{\partial T_\zeta'}{\partial t}-\frac{Rv_2}{c_0^2c_P}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial\zeta}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2T_\zeta'\right)+$$

$$\frac{\partial}{\partial t}\left(\frac{R}{\widetilde{p_{es}}\zeta+p_t}\int_0^\zeta\nabla\cdot\Big(\widetilde{p_{es}}V\Big)\mathrm{d}s\right)-\frac{\partial}{\partial t}\left(\frac{R\zeta}{\widetilde{p_{es}}\zeta+p_t}\nabla\widetilde{p_{es}}\cdot V\right)=$$

$$\frac{R\mu_2}{\widetilde{p_{es}}c_0^2c_P}\Delta T_t'+\frac{Rv_2}{c_0^2c_P}\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial T_t'}{\partial\zeta}\right)-\frac{R}{c_0^2}\frac{\partial}{\partial t}\left(\frac{R}{c_0^2}\kappa_aT'\right)-$$

$$\frac{R}{c_0^2}\frac{\partial}{\partial t}\left(\frac{RL^2}{c_0^2}\delta_{21}\delta_{22}\left(W(T)\left(\frac{\dot\zeta}{\widetilde{p_{es}}\zeta+p_t}\right)\right)\right),\tag{3.32}$$

将式(3.32)与$\widetilde{p_{es}}T_t'$做L^2内积,有

$$\frac{R}{2c_0^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\widetilde{p_{es}}\left|T_t'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+\int_\Omega\frac{Rv_2k_{s2}}{c_0^2c_P}\widetilde{p_{es}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left|T_{t\zeta}'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$\frac{R\mu_2}{c_0^2c_P}\int_\Omega\left|\nabla T_t'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta=$$

$$L_1+L_2+L_3+L_4+L_5,\tag{3.33}$$

经计算可得

$$L_1:=-\frac{R}{c_0^2}\int_\Omega\left(\Big(V^*\cdot\nabla\Big)T_t'-\left(\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot\Big(\widetilde{p_{es}}V^*\Big)\mathrm{d}s\right)T_{t\zeta}'\right)\cdot$$

$$\widetilde{p_{es}}T_t'\mathrm{d}\sigma\mathrm{d}\zeta=0,\tag{3.34}$$

类似式(3.26)的证明,可得

$$L_2:=-\frac{R}{c_0^2}\int_\Omega\left(\Big(V_t^*\cdot\nabla\Big)T'-\left(\frac{1}{\widetilde{p_{es}}}\int_0^\zeta\nabla\cdot\Big(\widetilde{p_{es}}V_t^*\Big)\mathrm{d}s\right)\frac{\partial T'}{\partial\zeta}\right)\cdot$$

$$\widetilde{p_{es}}T_t'\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|V_t\right\|_{L^2(\Omega)}^2+C\left\|T_t'\right\|_{L^2(\Omega)}^2+\varepsilon\left\|V_t\right\|_{H^1(\Omega)}^2+\varepsilon\left\|T_t'\right\|_{H^1(\Omega)}^2,\tag{3.35}$$

$$L_3:=-\int_\Omega\left(\frac{R}{\widetilde{p_{es}}\zeta+p_t}\int_0^\zeta\nabla\cdot\Big(\widetilde{p_{es}}V_t\Big)\mathrm{d}s\right)\widetilde{p_{es}}T_t'\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|T_t'\right\|_{L^2(\Omega)}^2+C\left\|V_t\right\|_{L^2(\Omega)}^2+\varepsilon\left\|\nabla V_t\right\|_{L^2(\Omega)}^2,\tag{3.36}$$

$$L_4:=\int_\Omega\left(\frac{R\zeta}{\widetilde{p_{es}}\zeta+p_t}\nabla\widetilde{p_{es}}\cdot V_t\right)\widetilde{p_{es}}T_t'\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C\left\|V_t\right\|_{L^2(\Omega)}^2+C\left\|T_t'\right\|_{L^2(\Omega)}^2,\tag{3.37}$$

$$L_5:=\frac{R}{c_0^2}\int_\Omega\left(\kappa_aT_t'-\frac{RL^2}{c_0^2}\delta_{21}\delta_{22}\left(\frac{\partial}{\partial t}\left(\frac{W(T)\dot\zeta}{\widetilde{p_{es}}\zeta+p_t}\right)\right)\right)\cdot$$

$$\widetilde{p_{es}}T_t'\mathrm{d}\sigma\mathrm{d}\zeta\leqslant C\left\|T_t'\right\|_{L^2(\Omega)}^2+C\left\|\nabla V_t\right\|_{L^2(\Omega)}\left\|T_t\right\|_{L^2(\Omega)}+$$

$$C\left\|\nabla V\right\|_{L^2(\Omega)}^{\frac{1}{2}}\left(\left\|\nabla V\right\|_{L^2(\Omega)}^{\frac{1}{2}}+\left\|\Delta V\right\|_{L^2(\Omega)}^{\frac{1}{2}}\right)\left\|T_t\right\|_{L^2(\Omega)}^{\frac{1}{2}}\cdot$$

$$\left(\left\|T_t\right\|_{L^2(\Omega)}^{\frac{1}{2}}+\left\|\nabla T_t\right\|_{L^2(\Omega)}^{\frac{1}{2}}\right)\left\|T_t\right\|_{L^2(\Omega)}\leqslant$$

$$C\left(1+\left\|\Delta V\right\|_{L^2(\Omega)}^2\right)\left\|T_t'\right\|_{L^2(\Omega)}^2+\varepsilon\left\|\nabla T_t'\right\|_{L^2(\Omega)}^2+\varepsilon\left\|\nabla V_t\right\|_{L^2(\Omega)}^2,\tag{3.38}$$

选取充分小的ε,由式(3.24~3.38)得

$$\frac{R}{2c_0^2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\widetilde{p_{\mathrm{es}}}\left|T_t'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+\frac{R\mu_2}{c_0^2c_P}\int_{\Omega}\left|\nabla T_t'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$\frac{Rv_2}{c_0^2c_P}\int_{\Omega}\widetilde{p_{\mathrm{es}}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left|T_{t\zeta}'\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C+C\left\|V_t\right\|_{L^2(\Omega)}^2+C\left\|T_t'\right\|_{L^2(\Omega)}^2+\varepsilon\left\|V_t\right\|_{H^1(\Omega)}^2,\quad(3.39)$$

类似可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\widetilde{p_{\mathrm{es}}}\left|q_t\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+\mu_3\int_{\Omega}\left|\nabla q_t\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$v_3\int_{\Omega}\widetilde{p_{\mathrm{es}}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left|q_{t\zeta}\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C+C\left\|V_t\right\|_{L^2(\Omega)}^2+C\left\|q_t\right\|_{L^2(\Omega)}^2,\quad(3.40)$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\widetilde{p_{\mathrm{es}}}\left|m_{wt}\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+\mu_4\int_{\Omega}\left|\nabla m_{wt}\right|^2\mathrm{d}\sigma\mathrm{d}\zeta+$$

$$v_4\int_{\Omega}\widetilde{p_{\mathrm{es}}}\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\left|m_{wt\zeta}\right|^2\mathrm{d}\sigma\mathrm{d}\zeta\leqslant$$

$$C+C\left\|V_t\right\|_{L^2(\Omega)}^2+C\left\|m_{wt}\right\|_{L^2(\Omega)}^2,\quad(3.41)$$

综合式(3.31、3.39~3.41),再利用Gronwall不等式,可以得到式(3.22)。

下面给出定理1的证明过程。首先证明$V\in L^{\infty}\left(\left[0,T\right];\ H^2(\Omega)\right)$,令

$$f:=-\frac{\partial V}{\partial t}-\left(V^*\cdot\nabla\right)V-\dot{\zeta}^*\frac{\partial V}{\partial\zeta}-$$

$$\left(2\omega\cos\theta+\frac{\cot\theta}{a}v_{\lambda}\right)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V-\frac{R\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}T'\nabla\widetilde{p_{\mathrm{es}}}+$$

$$\nabla\left(\frac{R\widetilde{T}_s}{\widetilde{p_{\mathrm{es}}}}\int_0^t\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}\overline{V}\right)\mathrm{d}\tau\right)+R\nabla\int_{\zeta}^1\frac{\widetilde{p_{\mathrm{es}}}T'(s)}{\widetilde{p_{\mathrm{es}}}s+p_t}\mathrm{d}s,\quad(3.42)$$

则有

$$-\frac{\mu_1}{\widetilde{p_{\mathrm{es}}}}\Delta V-v_1\frac{\partial}{\partial\zeta}\left(\left(\frac{g\zeta}{R\widetilde{T}}\right)^2\frac{\partial V}{\partial\zeta}\right)=f,\quad(3.43)$$

因为$V_{\zeta}\in L^{\infty}\left(\left[0,T\right];\ L^4(\Omega)\right)$,$V\in L^{\infty}\left(\left[0,T\right];\ H^1(\Omega)\right)$,$V_t\in L^{\infty}\left(\left[0,\mathrm{T}\right];\ L^2(\Omega)\right)$,则有:

$$\left\|\frac{\partial V}{\partial t}\right\|_{L^{\frac{4}{3}}(\Omega)}+\left\|\left(V^*\cdot\nabla\right)V\right\|_{L^{\frac{4}{3}}(\Omega)}+$$

$$\left\|-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^{\zeta}\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right)V_{\zeta}\right\|_{L^{\frac{4}{3}}(\Omega)}\leqslant$$

$$C\left\|V_t\right\|_{L^2(\Omega)}+C\left\|V^*\right\|_{L^4(\Omega)}\left\|\nabla V\right\|_{L^2(\Omega)}+$$

$$C\left\|\int_0^{\zeta}\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right\|_{L^2(\Omega)}\left\|V_{\zeta}\right\|_{L^4(\Omega)}\leqslant C,\quad(3.44)$$

$$\left\|\left(2\omega\cos\theta+\frac{\cot\theta}{a}v_{\lambda}\right)\begin{pmatrix}0&-1\\1&0\end{pmatrix}V\right\|_{L^{\frac{4}{3}}(\Omega)}+$$

$$\left\|\frac{R\zeta}{\widetilde{p_{\mathrm{es}}}\zeta+p_t}T'\nabla\widetilde{p_{\mathrm{es}}}\right\|_{L^{\frac{4}{3}}(\Omega)}+\left\|R\nabla\int_{\zeta}^1\frac{\widetilde{p_{\mathrm{es}}}T'(s)}{\widetilde{p_{\mathrm{es}}}s+p_t}\mathrm{d}s\right\|_{L^{\frac{4}{3}}(\Omega)}\leqslant$$

$$C\left\|V\right\|_{L^4(\Omega)}^2+C\left\|T'\right\|_{L^2(\Omega)}+C\left\|\nabla T'\right\|_{L^2(\Omega)}\leqslant C,\quad(3.45)$$

$$\left\|\nabla\left(\frac{R\widetilde{T}_s}{\widetilde{p_{\mathrm{es}}}}\int_0^t\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}\overline{V}\right)\mathrm{d}\tau\right)\right\|_{L^{\frac{4}{3}}(S^2)}\leqslant$$

$$\left\|\nabla\left(\frac{R\widetilde{T}_s}{\widetilde{p_{\mathrm{es}}}}\right)\int_0^t\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}\overline{V}\right)\mathrm{d}\tau\right\|_{L^2(S^2)}+$$

$$\left\|\frac{R\widetilde{T}_s}{\widetilde{p_{\mathrm{es}}}}\int_0^t\nabla\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}\overline{V}\right)\mathrm{d}\tau\right\|_{L^2(S^2)}\leqslant C,\quad(3.46)$$

所以有$f\in L^{\infty}\left(\left[0,T\right];\ L^{\frac{4}{3}}(\Omega)\right)$,由椭圆方程的正则性有$V\in L^{\infty}\left(\left[0,T\right];\ W^{2,\frac{4}{3}}(\Omega)\right)$,根据嵌入定理$W^{2,\frac{4}{3}}(\Omega)\subset W^{1,\frac{12}{5}}(\Omega)$,所以可得$V\in L^{\infty}\left(\left[0,T\right];\ W^{1,\frac{12}{5}}(\Omega)\right)$。再由$V_{\zeta}\in L^{\infty}\left(\left[0,\ T\right];L^4(\Omega)\right)$,$V\in L^{\infty}\left(\left[0,T\right];\ W^{1,\frac{12}{5}}(\Omega)\right)$,可得

$$\left\|-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^{\zeta}\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right)V_{\zeta}\right\|_{L^{\frac{3}{2}}(\Omega)}\leqslant$$

$$C\left\|V\right\|_{W^{1,\frac{12}{5}}(\Omega)}\left\|V_{\zeta}\right\|_{L^4(\Omega)}\leqslant C,\quad(3.47)$$

对于f中的其余项也可做类似的估计,进而可得$f\in L^{\infty}\left(\left[0,T\right];\ L^{\frac{3}{2}}(\Omega)\right)$,由椭圆方程的正则性理论有$V\in L^{\infty}\left(\left[0,T\right];\ W^{2,\frac{3}{2}}(\Omega)\right)$,再根据嵌入定理有$V\in L^{\infty}\left(\left[0,T\right];\ W^{1,3}(\Omega)\right)$。又因为$V_{\zeta}\in L^{\infty}\left(\left[0,\ T\right];L^4(\Omega)\right)$,$V\in L^{\infty}\left(\left[0,T\right];\ W^{1,3}(\Omega)\right)$,则有

$$\left\|-\left(\frac{1}{\widetilde{p_{\mathrm{es}}}}\int_0^{\zeta}\nabla\cdot\left(\widetilde{p_{\mathrm{es}}}V^*\right)\mathrm{d}s\right)V_{\zeta}\right\|_{L^{\frac{12}{7}}(\Omega)}\leqslant$$

$$C\left\|V\right\|_{W^{1,3}(\Omega)}\left\|V_{\zeta}\right\|_{L^4(\Omega)}\leqslant C,\quad(3.48)$$

进而可得$f\in L^{\infty}\left([0,T];\ L^{\frac{12}{7}}(\Omega)\right)$,根据椭圆方程的正则性理论有$V\in L^{\infty}\left([0,T];\ W^{2,\frac{12}{7}}(\Omega)\right)$,再由嵌入定理有$V\in L^{\infty}\left([0,T];\ W^{1,4}(\Omega)\right)$。再由$V_{\zeta}\in L^{\infty}\left([0,T];\ L^{4}(\Omega)\right)$,$V\in L^{\infty}\left([0,T];\ W^{1,4}(\Omega)\right)$,有

$$\left\|-\left(\frac{1}{\widetilde{p}_{\mathrm{es}}}\int_{0}^{\zeta}\nabla\cdot\left(\widetilde{p}_{\mathrm{es}}V^{*}\right)\mathrm{d}s\right)V_{\zeta}\right\|_{L^{2}(\Omega)}\leqslant C\|V\|_{W^{1,4}(\Omega)}\|V_{\zeta}\|_{L^{4}(\Omega)}\leqslant C,\tag{3.49}$$

进而可得$f\in L^{\infty}\left([0,T];\ L^{2}(\Omega)\right)$,根据椭圆方程的正则性理论有$V\in L^{\infty}\left([0,T];\ H^{2}(\Omega)\right)$。再利用迭代方法和椭圆方程的正则性理论可以证明$T',q,m_{w}\in L^{\infty}\left([0,T];\ H^{2}(\Omega)\right)$。

此外,利用文献[23-24]中证明强解唯一性的方法,亦可以证明该二阶正则强解是唯一的。

4 总结与展望

本论文研究了加入地形和水汽相变等因素的湿大气动力学方程组的初边值问题,在初值具有二阶正则性时,采用能量估计和迭代的方法证明了二阶正则整体强解的存在唯一性。新的困难点主要在于:(1)消除整层无辐散近似在方程中引入了一个新的表征可压缩性质的项$\nabla\left(\frac{R\widetilde{T}_{s}}{\widetilde{p}_{\mathrm{es}}}\int_{0}^{t}\nabla\cdot\left(\widetilde{p}_{\mathrm{es}}\overline{V}\right)\mathrm{d}\tau\right)$;(2)将外源强迫项假设为大气温度和压力等变量的非线性函数。利用Hölder不等式和Young不等式等就可以得到含有这些项的估计。未来将进一步考虑包含部分耗散的湿大气动力学方程组初边值问题强解的存在唯一性等。

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