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三维无压Euler-Navier-Stokes方程组的格林函数*

李海梁[1,2**],张 越[1]

(1. 首都师范大学数学科学学院,北京 100048;2. 首都师范大学交叉科学研究院,北京 100048)

摘要:本文研究三维无压Euler-Navier-Stokes耦合模型解的时空逐点行为,该模型可用于描述两流体运动。首先证明线性化系统的格林函数由惠更斯波、扩散波、Riesz波和包含由无压结构产生的稳态delta波的奇异部分组成,进而当初值具有适当的空间衰减率时得到线性化系统Cauchy问题整体解的时空逐点估计。

关键词:无压Euler-Navier-Stokes方程组;格林函数;时空逐点行为
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The Green's function of the pressureless Euler-Navier-Stokes system in $\mathbf{R}^3$*

LI Hailiang[1,2**], ZHANG Yue[1]

(1. School of Mathematical Sciences, Capital Normal University, Beijing 100048; 2. Academy for Multidisciplinary Studies, Capital Normal University, Beijing 100048)

Abstract: In this paper, we investigate the space-time pointwise behaviors of global solutions to the pressureless Euler-Navier-Stokes system in three dimensions, which characterizes the motion of a two-phase fluid. First, we construct the Green's function for the linearized system, which consists of the Huygens wave, the diffusion wave, the Riesz wave, and the singular part containing the stationary delta wave arising from the pressureless structure. Then, under the assumptions on the initial data with appropriate spatial decay rates, we obtain the space-time pointwise estimates of global solutions to the Cauchy problem for the linearized system.

Keywords: pressureless Euler-Navier-Stokes system; Green's function; space-time pointwise behavior
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0 Introduction

We consider the three-dimensional pressureless Euler-Navier-Stokes (pressureless Euler-NS) system for two fluids, which can be formally derived from the kinetic-fluid model, the so-called Vlasov-Navier-

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** 通信作者:hailiang_li@cnu.edu.cn

Stokes system, by assuming that the particle distribution is mono-kinetic, cf.[1-2].

The pressureless Euler-NS system consists of the pressureless Euler equations coupled with the compressible Navier-Stokes equations through a drag force term, given by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) = -\rho(u - v), \\ \partial_t n + \operatorname{div}(nv) = 0, \\ \partial_t (nv) + \operatorname{div}(nv \otimes v) + \nabla P(n) = \\ \qquad \mu \Delta v + (\mu + \nu)\nabla \operatorname{div} v + \rho(u - v), \end{cases} \tag{1}$$

where $\rho = \rho(x,t) > 0$ and $n = n(x,t) > 0$ are fluid densities, $u = u(x,t)$ and $v = v(x,t)$ are fluid velocities for $(x,t) \in \mathbf{R}^3 \times \mathbf{R}_+$. The pressure $P = P(n)$ satisfies $P'(n) > 0$ for $n > 0$. The constants μ and ν are the viscosity coefficients satisfying $\mu > 0$ and $\nu + 2\mu > 0$. Without loss of generality, we take $\mu = 1$ and $\nu = -1$ throughout this paper.

There is significant progress been made recently on the global existence and large-time behaviors of solutions to the pressureless Euler-NS system. Among them, the global existence and exponential time-decay rates of classical solutions to the initial value problem in the periodic domain $\mathbf{T}^3$ were obtained by Choi and Kwon [1]. Later, Guo et al[2] established the global well-posedness of classical solutions in $\mathbf{R}^3$ and proved that the solution converges to a constant equilibrium state at an algebraic rate with time. The blowup of classical solutions to the Cauchy problem was shown by Choi [3] and Dong et al[4] under some restrictions on the initial data.

For the pressureless Euler equations coupled with the incompressible Navier-Stokes equations, Ha et al[5] investigated the global well-posedness and large-time behavior of classical solutions in the periodic domain $\mathbf{T}^3$. Subsequently, Choi and Jung[6] proved the global existence and uniqueness of classical solutions in $\mathbf{R}^3$. The time-decay rates were also obtained in reference [6] and further improved in reference [7]. Besides, Zhai et al[8] considered this system in the critical homogeneous Besov space and obtained the optimal time-decay rates of the solutions.

Nevertheless, all these results can only describe the dissipative properties of the global solutions. The main purpose of the present paper is to study the space-time pointwise behaviors of solutions to the pressureless Euler-NS system (1) so as to investigate the wave propagation phenomenon for the two-phase fluid motion. To this end, inspired by the pioneering work [9-17] on the compressible Navier-Stokes system, we study the pointwise description of the Green's function and establish the space-time pointwise estimates of global solutions to the Cauchy problem for the linearized pressureless Euler-NS system.

We denote by $\phi = \rho u$ and $\psi = nv$ the momentum densities and define the new variables

$$\tilde{\rho} = \rho - \rho_*, \ \tilde{n} = n - n_*, \tag{2}$$

with the positive constants $\rho_* > 0$ and $n_* > 0$. Denote $c_0 = \sqrt{P'(n_*)}$, then the Cauchy problem for the linearized pressureless Euler-NS system can be written in terms of $(\tilde{\rho}, \phi, \tilde{n}, \psi)$ as

$$\begin{cases} \partial_t \tilde{\rho} + \operatorname{div}\phi = 0, \\ \partial_t \phi + \phi - \dfrac{\rho_*}{n_*}\psi = 0, \\ \partial_t \tilde{n} + \operatorname{div}\psi = 0, \\ \partial_t \psi + c_0^2 \nabla \tilde{n} - \phi + \dfrac{\rho_*}{n_*}\psi - \dfrac{1}{n_*}\Delta\psi = 0, \\ (\tilde{\rho}, \phi, \tilde{n}, \psi)(x,0) = (\tilde{\rho}_0, \phi_0, \tilde{n}_0, \psi_0)(x). \end{cases} \tag{3}$$

Let $G = G(x,t)$ be the Green's function generated by the system (3), satisfying

$$\begin{cases} \partial_t G - \boldsymbol{L} G = 0, \\ G(x,0) = \delta(x)\boldsymbol{I}_8, \end{cases} \tag{4}$$

where $\delta(x)$ represents the standard Dirac delta function, and the operator $\boldsymbol{L}$ is defined by

$$\boldsymbol{L} = \begin{pmatrix} 0 & -\operatorname{div} & 0 & 0 \\ 0 & -\boldsymbol{I}_3 & 0 & \dfrac{\rho_*}{n_*}\boldsymbol{I}_3 \\ 0 & 0 & 0 & -\operatorname{div} \\ 0 & \boldsymbol{I}_3 & -c_0^2\nabla & -\dfrac{\rho_*}{n_*}\boldsymbol{I}_3 + \dfrac{1}{n_*}\Delta \end{pmatrix} \tag{5}$$

with the $m \times m$ identity matrix $\boldsymbol{I}_m$ for $m > 0$ being a positive integer.

First, we have the following results on the Green's function.

Theorem 1 Let $G = (G_{jk})_{j,k=1,2,3,4}(x,t)$ be the

Green's function to (4), and α be a multi-index with $|\alpha| \geqslant 0$. Then, there exists a constant $C_0 > 0$ such that

$$\left|\partial_x^\alpha (G - G_S)_{j1}(x,t)\right| = 0,\ j = 1,2,3,4, \tag{6}$$

$$\begin{aligned}&\left|\partial_x^\alpha (G - G_S)_{1k}(x,t)\right| + \left|\partial_x^\alpha (G - G_S)_{3k}(x,t)\right| + \\ &\left|\partial_x^\alpha (G - G_S)_{k3}(x,t)\right| \leqslant \\ &C_0 (1+t)^{-\frac{4+|\alpha|}{2}} \mathrm{e}^{-\frac{(|x|-c_* t)^2}{C_0(1+t)}} + C_0 \mathrm{e}^{-\frac{t+|x|}{C_0}} + \\ &C_0 t^{-\frac{3+|\alpha|}{2}} \mathrm{e}^{-\frac{t}{C_0}} \mathrm{e}^{-\frac{|x|^2}{C_0 t}},\ k = 2,3,4,\end{aligned} \tag{7}$$

$$\begin{aligned}&\left|\partial_x^\alpha (G - G_S)_{22}(x,t)\right| + \left|\partial_x^\alpha (G - G_S)_{24}(x,t)\right| + \\ &\left|\partial_x^\alpha (G - G_S)_{42}(x,t)\right| + \left|\partial_x^\alpha (G - G_S)_{44}(x,t)\right| \leqslant \\ &C_0 (1+t)^{-\frac{4+|\alpha|}{2}} \mathrm{e}^{-\frac{(|x|-c_* t)^2}{C_0(1+t)}} + C_0 (1+t)^{-\frac{3+|\alpha|}{2}} \mathrm{e}^{-\frac{|x|^2}{C_0(1+t)}} + \\ &C_0 (1+t)^{-\frac{3+|\alpha|}{2}} \chi_{\{|x| \leqslant c_*(1+t)\}} B_{\frac{3+|\alpha|}{2}}(|x|,t) + \\ &C_0 \mathrm{e}^{-\frac{t+|x|}{C_0}} + C_0 t^{-\frac{3+|\alpha|}{2}} \mathrm{e}^{-\frac{t}{C_0}} \mathrm{e}^{-\frac{|x|^2}{C_0 t}},\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\left|\partial_x^\alpha ((G - G_S)_{j2} - (G - G_S)_{j4})(x,t)\right| \leqslant \\ &C_0 (1+t)^{-\frac{5+|\alpha|}{2}} \mathrm{e}^{-\frac{(|x|-c_* t)^2}{C_0(1+t)}} + C_0 \mathrm{e}^{-\frac{t+|x|}{C_0}} + \\ &C_0 t^{-\frac{3+|\alpha|}{2}} \mathrm{e}^{-\frac{t}{C_0}} \mathrm{e}^{-\frac{|x|^2}{C_0 t}},\ j = 1,2,3,4,\end{aligned} \tag{9}$$

where $c_* = c_0 \sqrt{\dfrac{n_*}{\rho_* + n_*}}$ is a positive constant, $B_a(|x|,t) := \left(1 + \dfrac{|x|^2}{1+t}\right)^{-a}$ for the constant $a > 0$, and χ_A is the characteristic function for the set A satisfying

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The singular part $G_S = G_S(x,t)$ of the Green's function can be decomposed into

$$G_S(x,t) = G_0(x) + G_1(x,t), \tag{11}$$

where G_0 is the singular stationary wave and G_1 is the singular leading short wave, defined by (19) and (35) respectively.

With the help of Theorem 1, we then obtain the space-time pointwise estimates of global solutions to the Cauchy problem (3).

Theorem 2 Assume that the initial data $(\tilde{\rho}_0, \phi_0, \tilde{n}_0, \psi_0)$, for the constant $\theta > \dfrac{21}{10}$, satisfies

$$\left|(\tilde{\rho}_0, \phi_0, \partial_x \phi_0, \tilde{n}_0, \psi_0)(x)\right| \leqslant (1 + |x|^2)^{-\theta}. \tag{12}$$

There exists a constant $\bar{C} > 0$ such that the Cauchy problem (3) admits a unique global solution $(\tilde{\rho}, \phi, \tilde{n}, \psi)$ satisfying

$$\begin{aligned}&\left|\tilde{\rho}(x,t)\right| \leqslant \bar{C}(1+|x|^2)^{-\theta} + \\ &\bar{C}(1+t)^{-2}\left(B_{\frac{3}{2}}(|x| - c_* t, t) + B_{\frac{3}{2}}(|x|,t)\right), \\ &\left|\phi(x,t)\right| \leqslant \\ &\bar{C}(1+t)^{-\frac{3}{2}}\left((1+t)^{-\frac{1}{2}} B_{\frac{3}{2}}(|x| - c_* t, t) + B_{\frac{3}{2}}(|x|,t)\right), \\ &\left|\tilde{n}(x,t)\right| \leqslant \bar{C}(1+t)^{-2}\left(B_{\frac{3}{2}}(|x| - c_* t, t) + B_{\frac{3}{2}}(|x|,t)\right), \\ &\left|\psi(x,t)\right| \leqslant \\ &\bar{C}(1+t)^{-\frac{3}{2}}\left((1+t)^{-\frac{1}{2}} B_{\frac{3}{2}}(|x| - c_* t, t) + B_{\frac{3}{2}}(|x|,t)\right),\end{aligned} \tag{13}$$

where $B_a(|x|,t) := \left(1 + \dfrac{|x|^2}{1+t}\right)^{-a}$ for the constants $a > 0$ and $c_* = c_0 \sqrt{\dfrac{n_*}{\rho_* + n_*}}$.

In addition, if the initial data further satisfies

$$\tilde{\rho}_0 - \mathrm{div}\phi_0 - \frac{\rho_*}{n_*} \tilde{n}_0 = 0, \tag{14}$$

then it holds for $\tilde{\rho}(x,t)$ that

$$\left|\tilde{\rho}(x,t)\right| \leqslant \bar{C}(1+t)^{-2}\left(B_{\frac{3}{2}}(|x| - c_* t, t) + B_{\frac{3}{2}}(|x|,t)\right). \tag{15}$$

The rest parts of this paper are arranged as follows. In Section 1, some preliminary calculations on the Green's function in the Fourier space are given. In Section 2, we obtain the pointwise estimates of the Green's function inside the finite Mach number region by the long wave and short wave decomposition. In Section 3, we apply the weighted energy method to establish the pointwise estimates of the Green's function outside the finite Mach number region. Finally, we investigate the space-time pointwise behaviors of global solutions to the Cauchy problem of the linearized pressureless Euler-NS system in Section 4.

1 Spectral analysis

Taking the Fourier transform of system (4) with respect to x, we have

$$\begin{cases}\partial_t \hat{G} - \hat{\boldsymbol{L}}(\boldsymbol{\xi})\hat{G} = 0,\\ \hat{G}(\boldsymbol{\xi}, 0) = \boldsymbol{I}_8,\end{cases} \tag{16}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}$, and $\hat{\boldsymbol{L}}(\boldsymbol{\xi})$ is defined as

$$\hat{\boldsymbol{L}}(\boldsymbol{\xi}) = \begin{pmatrix} 0 & -\mathrm{i}\boldsymbol{\xi}^{\mathrm{T}} & 0 & 0\\ 0 & -\boldsymbol{I}_3 & 0 & \dfrac{\rho_*}{n_*}\boldsymbol{I}_3\\ 0 & 0 & 0 & -\mathrm{i}\boldsymbol{\xi}^{\mathrm{T}}\\ 0 & \boldsymbol{I}_3 & -\mathrm{i}c_0^2\boldsymbol{\xi} & -(\dfrac{\rho_*}{n_*}+\dfrac{1}{n_*}|\boldsymbol{\xi}|^2)\boldsymbol{I}_3 \end{pmatrix}.$$

The eigenvalues of the matrix $\hat{\boldsymbol{L}}$ are computed from the determinant

$$\det(\lambda\boldsymbol{I}_8 - \hat{\boldsymbol{L}}) = \frac{1}{n_*}\lambda\left(n_*\lambda^2 + (\rho_* + n_* + |\boldsymbol{\xi}|^2)\lambda + |\boldsymbol{\xi}|^2\right)^2 H(\lambda, |\boldsymbol{\xi}|) = 0 \tag{17}$$

with

$$H(\lambda, |\boldsymbol{\xi}|) = n_*\lambda^3 + (\rho_* + n_* + |\boldsymbol{\xi}|^2)\lambda^2 + (c_0^2 n_* + 1)|\boldsymbol{\xi}|^2\lambda + c_0^2 n_*|\boldsymbol{\xi}|^2,$$

which gives rise to

$$\lambda_0 = 0,$$

$$\lambda_1 = -\frac{\rho_* + n_* + |\boldsymbol{\xi}|^2}{2n_*} + \frac{\sqrt{\left(\rho_* + n_* + |\boldsymbol{\xi}|^2\right)^2 - 4n_*|\boldsymbol{\xi}|^2}}{2n_*}, \text{(double)}$$

$$\lambda_2 = -\frac{\rho_* + n_* + |\boldsymbol{\xi}|^2}{2n_*} - \frac{\sqrt{\left(\rho_* + n_* + |\boldsymbol{\xi}|^2\right)^2 - 4n_*|\boldsymbol{\xi}|^2}}{2n_*}, \text{(double)}$$

and λ_j satisfies $H(\lambda_j, |\boldsymbol{\xi}|) = 0$ for $j = 3, 4, 5$. Since the appearance of the zero eigenvalue $\lambda_0 = 0$, we decompose the semigroup $\hat{G}(\boldsymbol{\xi}, t)$ into two parts as follows

$$\hat{G}(\boldsymbol{\xi}, t) = \hat{G}_0(\boldsymbol{\xi}) + \hat{\varsigma}(\boldsymbol{\xi}, t), \tag{18}$$

where $\hat{G}_0(\boldsymbol{\xi})$ is the project operator corresponding to λ_0, given by

$$\hat{G}_0(\boldsymbol{\xi}) = \begin{pmatrix} 1 & -\mathrm{i}\boldsymbol{\xi}^{\mathrm{T}} & -\dfrac{\rho_*}{n_*} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

and the elements of $\hat{\varsigma} = (\hat{\varsigma}_{jk})_{j,k=1,2,3,4}(\boldsymbol{\xi}, t)$ satisfy

$$\hat{\varsigma}_{j1} = 0,\ j = 1, 2, 3, 4,$$

$$\hat{\varsigma}_{12} = \left\{\frac{(1+\lambda_4)(1+\lambda_5)\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)} + \frac{(1+\lambda_3)(1+\lambda_5)\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)} + \frac{(1+\lambda_3)(1+\lambda_4)\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\mathrm{i}\boldsymbol{\xi}^{\mathrm{T}},$$

$$\hat{\varsigma}_{13} = -\frac{\rho_* c_0^2|\boldsymbol{\xi}|^2}{n_*}\left\{\frac{\mathrm{e}^{\lambda_3 t}}{\lambda_3(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)} + \frac{\mathrm{e}^{\lambda_4 t}}{\lambda_4(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)} + \frac{\mathrm{e}^{\lambda_5 t}}{\lambda_5(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\},$$

$$\hat{\varsigma}_{14} = -\frac{\rho_*}{n_*}\left\{\frac{\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)} + \frac{\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)} + \frac{\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\mathrm{i}\boldsymbol{\xi}^{\mathrm{T}},$$

$$\hat{\varsigma}_{22} = \frac{(1+\lambda_1)\mathrm{e}^{\lambda_2 t} - (1+\lambda_2)\mathrm{e}^{\lambda_1 t}}{\lambda_1-\lambda_2}\boldsymbol{I}_3 - \left\{\frac{(1+\lambda_1)\mathrm{e}^{\lambda_2 t} - (1+\lambda_2)\mathrm{e}^{\lambda_1 t}}{\lambda_1-\lambda_2} + \frac{\lambda_3(1+\lambda_4)(1+\lambda_5)\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)} + \frac{\lambda_4(1+\lambda_3)(1+\lambda_5)\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)} + \frac{\lambda_5(1+\lambda_3)(1+\lambda_4)\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\frac{\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}}{|\boldsymbol{\xi}|^2},$$

$$\hat{\varsigma}_{23} = -\frac{\rho_* c_0^2}{n_*}\left\{\frac{\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)} + \frac{\mathrm{e}^{\lambda_4 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)} + \frac{\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\mathrm{i}\boldsymbol{\xi},$$

$$\hat{\varsigma}_{24} = \frac{\rho_*(\mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t})}{n_*(\lambda_1-\lambda_2)}\boldsymbol{I}_3 + \frac{\rho_*}{n_*}\left\{-\frac{\mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t}}{\lambda_1-\lambda_2} + \frac{\lambda_3\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)} + \frac{\lambda_4\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)} + \frac{\lambda_5\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\frac{\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}}{|\boldsymbol{\xi}|^2},$$

$$\hat{\varsigma}_{32}=-\left\{\frac{\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)}+\frac{\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)}+\frac{\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\mathrm{i}\xi^{\mathrm{T}},$$

$$\hat{\varsigma}_{33}=-c_0^2|\xi|^2\left\{\frac{(1+\lambda_3)\mathrm{e}^{\lambda_3 t}}{\lambda_3(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)}+\frac{(1+\lambda_4)\mathrm{e}^{\lambda_4 t}}{\lambda_4(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)}+\frac{(1+\lambda_5)\mathrm{e}^{\lambda_5 t}}{\lambda_5(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\},$$

$$\hat{\varsigma}_{34}=-\left\{\frac{(1+\lambda_3)\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)}+\frac{(1+\lambda_4)\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)}+\frac{(1+\lambda_5)\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\mathrm{i}\xi^{\mathrm{T}},$$

$$\hat{\varsigma}_{42}=\frac{\mathrm{e}^{\lambda_1 t}-\mathrm{e}^{\lambda_2 t}}{\lambda_1-\lambda_2}\boldsymbol{I}_3+\left\{-\frac{\mathrm{e}^{\lambda_1 t}-\mathrm{e}^{\lambda_2 t}}{\lambda_1-\lambda_2}+\frac{\lambda_3\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)}+\frac{\lambda_4\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)}+\frac{\lambda_5\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2},$$

$$\hat{\varsigma}_{43}=-c_0^2\left\{\frac{(1+\lambda_3)\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)}+\frac{(1+\lambda_4)\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)}+\frac{(1+\lambda_5)\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\mathrm{i}\xi,$$

$$\hat{\varsigma}_{44}=\frac{(1+\lambda_1)\mathrm{e}^{\lambda_1 t}-(1+\lambda_2)\mathrm{e}^{\lambda_2 t}}{\lambda_1-\lambda_2}\boldsymbol{I}_3+\left\{-\frac{(1+\lambda_1)\mathrm{e}^{\lambda_1 t}-(1+\lambda_2)\mathrm{e}^{\lambda_2 t}}{\lambda_1-\lambda_2}+\frac{\lambda_3(1+\lambda_3)\mathrm{e}^{\lambda_3 t}}{(\lambda_3-\lambda_4)(\lambda_3-\lambda_5)}+\frac{\lambda_4(1+\lambda_4)\mathrm{e}^{\lambda_4 t}}{(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)}+\frac{\lambda_5(1+\lambda_5)\mathrm{e}^{\lambda_5 t}}{(\lambda_5-\lambda_3)(\lambda_5-\lambda_4)}\right\}\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}.$$

2　Pointwise estimates inside the finite Mach region

In this section, we are going to investigate the pointwise behaviors of $\varsigma(x,t)$. To this end, we decompose $\hat{\varsigma}$ defined above in the Fourier space as

$$\hat{\varsigma}=\hat{\varsigma}^l+\hat{\varsigma}^m+\hat{\varsigma}^h:=\left(\chi_{\{|\xi|\leqslant r\}}+\chi_{\{r\leqslant|\xi|\leqslant R\}}+\chi_{\{|\xi|\geqslant R\}}\right)\hat{\varsigma}$$

with the positive constants $r<1$ sufficiently small and $R>1$ sufficiently large.

It is direct to obtain the following asymptotic expansions of the eigenvalues.

Lemma 2.1　Let $\lambda_j(j=1,2,\cdots,5)$ be the eigenvalues given by (17), then we have that

(1) for $0<|\xi|\ll 1$, it holds

$$\lambda_1=-\eta|\xi|^2+O(|\xi|^4),\ (\text{double})$$

$$\lambda_2=-\frac{1}{\eta n_*}-\frac{\eta\rho_*}{n_*}|\xi|^2+O(|\xi|^4),\ (\text{double})$$

$$\lambda_3=-\frac{1}{\eta n_*}-\frac{\eta\rho_*(1-c_*^2 n_*)}{n_*}|\xi|^2+O(|\xi|^4),$$

$$\lambda_4=-\kappa|\xi|^2+\mathrm{i}c_*|\xi|+\mathrm{i}O(|\xi|^3),$$

$$\lambda_5=-\kappa|\xi|^2-\mathrm{i}c_*|\xi|+\mathrm{i}O(|\xi|^3),$$

(2) for $|\xi|\gg 1$, it holds

$$\lambda_1=-1+\rho_*|\xi|^{-2}+O(|\xi|^{-4}),\ (\text{double})$$

$$\lambda_2=-\frac{1}{n_*}|\xi|^2-\frac{\rho_*}{n_*}-\rho_*|\xi|^{-2}+O(|\xi|^{-4}),\ (\text{double})$$

$$\lambda_3=-1-\frac{\eta\rho_*}{c_*^2-\eta}|\xi|^{-2}+O(|\xi|^{-4}),$$

$$\lambda_4=-\frac{c_*^2}{\eta}+\frac{c_*^4(1-c_*^2 n_*)}{\eta^2(c_*^2-\eta)}|\xi|^{-2}+O(|\xi|^{-4}),$$

$$\lambda_5=-\frac{1}{n_*}|\xi|^2-\frac{\eta\rho_*-c_*^2 n_*}{\eta n_*}-\frac{\eta^2\rho_*+\eta c_*^2\rho_*-c_*^4 n_*}{\eta^2}|\xi|^{-2}+O(|\xi|^{-4}),$$

where $\eta=\dfrac{1}{\rho_*+n_*}$, $c_*=c_0\sqrt{\eta n_*}$, and $\kappa=\dfrac{1}{2}\eta(1+c_*^2\rho_*)$.

First, we consider the long wave ς^l corresponding to the low frequency $0<|\xi|\ll 1$ inside the finite Mach region $|x|\leqslant 2Mc_*(1+t)$ with a suitably large constant $M>1$ to be confirmed later. Note that when $0<|\xi|\ll 1$, the eigenvalues are analytic so it is reasonable to apply the complex variable technique. Indeed, we can derive the following sharp estimates.

Proposition 2.2　Let α be a multi-index with $|\alpha|\geqslant 0$. There exists a constant $C_1>0$ such that it

holds for the long wave $\varsigma^l=(\varsigma_{jk}^l)_{j,k=1,2,3,4}(x,t)$ that

$$\left|\partial_x^\alpha\varsigma_{j1}^l(x,t)\right|=0,\ j=1,2,3,4, \tag{20}$$

$$\left|\partial_x^\alpha(\varsigma_{1k}^l,\varsigma_{3k}^l,\varsigma_{k3}^l)(x,t)\right|\leqslant C_1(1+t)^{-\frac{4+|\alpha|}{2}}\mathrm{e}^{-\frac{(|x|-c_*t)^2}{C_1(1+t)}},\ k=2,3,4, \tag{21}$$

$$\left|\partial_x^\alpha(\varsigma_{22}^l,\varsigma_{24}^l,\varsigma_{42}^l,\varsigma_{44}^l)(x,t)\right|\leqslant C_1(1+t)^{-\frac{3+|\alpha|}{2}}\left\{(1+t)^{-\frac12}\mathrm{e}^{-\frac{(|x|-c_*t)^2}{C_1(1+t)}}+\mathrm{e}^{-\frac{|x|^2}{C_1(1+t)}}+\chi_{\{|x|\leqslant c_*(1+t)\}}B_{\frac{3+|\alpha|}{2}}(|x|,t)\right\}, \tag{22}$$

where $B_a(|x|,t):=\left(1+\dfrac{|x|^2}{1+t}\right)^{-a}$ for the constant $a>0$, and the characteristic function χ is defined by (10). In addition, one can deduce that

$$\left|\partial_x^\alpha(\varsigma_{j2}^l-\varsigma_{j4}^l)(x,t)\right|\leqslant C_1(1+t)^{-\frac{5+|\alpha|}{2}}\mathrm{e}^{-\frac{(|x|-c_*t)^2}{C_1(1+t)}},\ j=1,2,3,4. \tag{23}$$

Proof It follows from Lemma 2.1 and the explicit formulas for the elements of $\hat{\varsigma}$ that

$$\hat{\varsigma}_{j1}^l=0,\ j=1,2,3,4,$$

$$\hat{\varsigma}_{12}^l=-\mathrm{i}\eta\rho_*\mathrm{e}^{-\kappa|\xi|^2t}\left\{\frac{\sin(c_*|\xi|t)}{c_*|\xi|}\left[\cos(\sigma(|\xi|^2)|\xi|t)+\eta(\rho_*+2n_*)c_*|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]+\cos(c_*|\xi|t)\left[\frac{1}{c_*|\xi|}\sin(\sigma(|\xi|^2)|\xi|t)-\eta(\rho_*+2n_*)\cos(\sigma(|\xi|^2)|\xi|t)\right]\right\}\xi^{\mathrm{T}}+\mathrm{i}\eta^2n_*^2\mathrm{e}^{\lambda_3t}\xi^{\mathrm{T}},$$

$$\hat{\varsigma}_{13}^l=\frac{\rho_*}{n_*}\mathrm{e}^{-\kappa|\xi|^2t}\left\{\frac{\sin(c_*|\xi|t)}{c_*|\xi|}\left[-c_*|\xi|\sin(\sigma(|\xi|^2)|\xi|t)+\frac12\eta(1+c_*^2(\rho_*+2n_*))|\xi|^2\cos(\sigma(|\xi|^2)|\xi|t)\right]+\cos(c_*|\xi|t)\left[\cos(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2c_*}\eta(1+c_*^2(\rho_*+2n_*))|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]\right\},$$

$$\hat{\varsigma}_{14}^l=-\mathrm{i}\eta\rho_*\mathrm{e}^{-\kappa|\xi|^2t}\left\{\frac{\sin(c_*|\xi|t)}{c_*|\xi|}\left[\cos(\sigma(|\xi|^2)|\xi|t)+\eta n_*c_*|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]+\cos(c_*|\xi|t)\left[\frac{1}{c_*|\xi|}\sin(\sigma(|\xi|^2)|\xi|t)-\eta n_*\cos(\sigma(|\xi|^2)|\xi|t)\right]\right\}\xi^{\mathrm{T}}-\mathrm{i}\eta^2\rho_*n_*\mathrm{e}^{\lambda_3t}\xi^{\mathrm{T}},$$

$$\hat{\varsigma}_{22}^l=\eta\left(\rho_*\mathrm{e}^{\lambda_1t}+n_*\mathrm{e}^{\lambda_2t}\right)\boldsymbol{I}_3-\eta\left(\rho_*\mathrm{e}^{\lambda_1t}+n_*\mathrm{e}^{\lambda_2t}-n_*\mathrm{e}^{\lambda_3t}\right)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}+\eta\rho_*\mathrm{e}^{-\kappa|\xi|^2t}\times\left\{\frac{\sin(c_*|\xi|t)}{c_*|\xi|}\left[-c_*|\xi|\sin(\sigma(|\xi|^2)|\xi|t)+\frac12\eta(c_*^2(\rho_*+4n_*)-1)|\xi|^2\cos(\sigma(|\xi|^2)|\xi|t)\right]+\cos(c_*|\xi|t)\left[\cos(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2c_*}\eta(c_*^2(\rho_*+4n_*)-1)|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]\right\}\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2},$$

$$\hat{\varsigma}_{23}^l=-\mathrm{i}\frac{\rho_*c_*^2}{n_*}\mathrm{e}^{-\kappa|\xi|^2t}\left\{\frac{\sin(c_*|\xi|t)}{c_*|\xi|}\left[\cos(\sigma(|\xi|^2)|\xi|t)+\eta n_*c_*|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]+\cos(c_*|\xi|t)\left[\frac{1}{c_*|\xi|}\sin(\sigma(|\xi|^2)|\xi|t)-\eta n_*\cos(\sigma(|\xi|^2)|\xi|t)\right]\right\}\xi-\mathrm{i}\eta\rho_*c_*^2\mathrm{e}^{\lambda_3t}\xi,$$

$$\hat{\varsigma}_{24}^{l}=\eta\rho_{*}\left(\mathrm{e}^{\lambda_1 t}-\mathrm{e}^{\lambda_2 t}\right)\boldsymbol{I}_3-\eta\rho_{*}\left(\mathrm{e}^{\lambda_1 t}-\mathrm{e}^{\lambda_2 t}+\mathrm{e}^{\lambda_3 t}\right)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}+\eta\rho_{*}\mathrm{e}^{-\kappa|\xi|^2 t}\times$$
$$\left\{\frac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}\left[-c_{*}|\xi|\sin(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2}\eta(c_{*}^2(2n_{*}-\rho_{*})-1)|\xi|^2\cos(\sigma(|\xi|^2)|\xi|t)\right]+\right.$$
$$\left.\cos(c_{*}|\xi|t)\left[\cos(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2c_{*}}\eta(c_{*}^2(2n_{*}-\rho_{*})-1)|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]\right\}\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2},$$

$$\hat{\varsigma}_{32}^{l}=-\mathrm{i}\eta n_{*}\mathrm{e}^{-\kappa|\xi|^2 t}\left\{\frac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}\left[\cos(\sigma(|\xi|^2)|\xi|t)+\eta n_{*}c_{*}|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]+\right.$$
$$\left.\cos(c_{*}|\xi|t)\left[\frac{1}{c_{*}|\xi|}\sin(\sigma(|\xi|^2)|\xi|t)-\eta n_{*}\cos(\sigma(|\xi|^2)|\xi|t)\right]\right\}\xi^{\mathrm{T}}-\mathrm{i}\eta^2 n_{*}^2\mathrm{e}^{\lambda_3 t}\xi^{\mathrm{T}},$$

$$\hat{\varsigma}_{33}^{l}=\mathrm{e}^{-\kappa|\xi|^2 t}\left\{\frac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}\left[-c_{*}|\xi|\sin(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2}\eta(1-c_{*}^2\rho_{*})|\xi|^2\cos(\sigma(|\xi|^2)|\xi|t)\right]+\right.$$
$$\left.\cos(c_{*}|\xi|t)\left[\cos(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2c_{*}}\eta(1-c_{*}^2\rho_{*})|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]\right\},$$

$$\hat{\varsigma}_{34}^{l}=-\mathrm{i}\eta n_{*}\mathrm{e}^{-\kappa|\xi|^2 t}\left\{\frac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}\left[\cos(\sigma(|\xi|^2)|\xi|t)-\eta\rho_{*}c_{*}|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]+\right.$$
$$\left.\cos(c_{*}|\xi|t)\left[\frac{1}{c_{*}|\xi|}\sin(\sigma(|\xi|^2)|\xi|t)+\eta\rho_{*}\cos(\sigma(|\xi|^2)|\xi|t)\right]\right\}\xi^{\mathrm{T}}-\mathrm{i}\eta^2\rho_{*}n_{*}\mathrm{e}^{\lambda_3 t}\xi^{\mathrm{T}},$$

$$\hat{\varsigma}_{42}^{l}=\eta n_{*}\left(\mathrm{e}^{\lambda_1 t}-\mathrm{e}^{\lambda_2 t}\right)\boldsymbol{I}_3-\eta n_{*}\left(\mathrm{e}^{\lambda_1 t}-\mathrm{e}^{\lambda_2 t}+\mathrm{e}^{\lambda_3 t}\right)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}+\eta n_{*}\mathrm{e}^{-\kappa|\xi|^2 t}\times$$
$$\left\{\frac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}\left[-c_{*}|\xi|\sin(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2}\eta(c_{*}^2(2n_{*}-\rho_{*})-1)|\xi|^2\cos(\sigma(|\xi|^2)|\xi|t)\right]+\right.$$
$$\left.\cos(c_{*}|\xi|t)\left[\cos(\sigma(|\xi|^2)|\xi|t)+\frac{1}{2c_{*}}\eta(c_{*}^2(2n_{*}-\rho_{*})-1)|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]\right\}\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2},$$

$$\hat{\varsigma}_{43}^{l}=-\mathrm{i}c_{*}^2\mathrm{e}^{-\kappa|\xi|^2 t}\left\{\frac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}\left[\cos(\sigma(|\xi|^2)|\xi|t)-\eta\rho_{*}c_{*}|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]+\right.$$
$$\left.\cos(c_{*}|\xi|t)\left[\frac{1}{c_{*}|\xi|}\sin(\sigma(|\xi|^2)|\xi|t)+\eta\rho_{*}\cos(\sigma(|\xi|^2)|\xi|t)\right]\right\}\xi-\mathrm{i}\eta\rho_{*}c_{*}^2\mathrm{e}^{\lambda_3 t}\xi,$$

$$\hat{\varsigma}_{44}^{l}=\eta\left(n_{*}\mathrm{e}^{\lambda_1 t}+\rho_{*}\mathrm{e}^{\lambda_2 t}\right)\boldsymbol{I}_3-\eta\left(n_{*}\mathrm{e}^{\lambda_1 t}+\rho_{*}\mathrm{e}^{\lambda_2 t}-\rho_{*}\mathrm{e}^{\lambda_3 t}\right)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}+\eta n_{*}\mathrm{e}^{-\kappa|\xi|^2 t}\times$$
$$\left\{\frac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}\left[-c_{*}|\xi|\sin(\sigma(|\xi|^2)|\xi|t)-\frac{1}{2}\eta(3c_{*}^2\rho_{*}+1)|\xi|^2\cos(\sigma(|\xi|^2)|\xi|t)\right]+\right.$$
$$\left.\cos(c_{*}|\xi|t)\left[\cos(\sigma(|\xi|^2)|\xi|t)-\frac{1}{2c_{*}}\eta(3c_{*}^2\rho_{*}+1)|\xi|\sin(\sigma(|\xi|^2)|\xi|t)\right]\right\}\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2},$$

where $\sigma(|\xi|^2)$ is an analytic function in $|\xi|^2$ and $\sigma(0)=0$. Due to the singularity for $|\xi|\ll 1$, we can take $\hat{\varsigma}_{44}^{l}$ into account and rewrite it as the following equations.

$$\begin{aligned}\hat{\varsigma}_{44}^{l} = {} & \eta n_{*}\Big\{\mathrm{e}^{-\eta|\xi|^{2}t+O(1)|\xi|^{4}t}\boldsymbol{I}_{3}+\\ & \hat{\omega}\mathrm{e}^{-\kappa|\xi|^{2}t}\left[-c_{*}\frac{\sin(\sigma(|\xi|^{2})|\xi|t)}{\sigma(|\xi|^{2})|\xi|t}\sigma(|\xi|^{2})t-\frac{1}{2}\eta(3c_{*}^{2}\rho_{*}+1)\cos(\sigma(|\xi|^{2})|\xi|t)\right]\xi\xi^{\mathrm{T}}+\\ & \hat{\omega}_{t}\mathrm{e}^{-\kappa|\xi|^{2}t}\left[-\frac{1}{2c_{*}}\eta(3c_{*}^{2}\rho_{*}+1)\frac{\sin(\sigma(|\xi|^{2})|\xi|t)}{\sigma(|\xi|^{2})|\xi|t}\sigma(|\xi|^{2})t\right]\xi\xi^{\mathrm{T}}+\\ & (\cos(c_{*}|\xi|t)-1)\mathrm{e}^{-\kappa|\xi|^{2}t}\cos(\sigma(|\xi|^{2})|\xi|t)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^{2}}+\left(\mathrm{e}^{-\kappa|\xi|^{2}t}-\mathrm{e}^{-\eta|\xi|^{2}t+O(1)|\xi|^{4}t}\right)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^{2}}+\\ & \cos(c_{*}|\xi|t)\left[\cos(\sigma(|\xi|^{2})|\xi|t)-\frac{1}{2c_{*}}\eta(3c_{*}^{2}\rho_{*}+1)|\xi|\sin(\sigma(|\xi|^{2})|\xi|t)\right]\Big\}\frac{\xi\xi^{\mathrm{T}}}{|\xi|^{2}}+\\ & \mathrm{e}^{-\frac{1}{\eta n_{*}}t}\left(O(1)|\xi|^{2}t\boldsymbol{I}_{3}+O(1)t\xi\xi^{\mathrm{T}}\right),\end{aligned}\tag{24}$$

where $\hat{\omega}(\xi,t)=\dfrac{\sin(c_{*}|\xi|t)}{c_{*}|\xi|}$ and $\hat{\omega}_{t}(\xi,t)=\cos(c_{*}|\xi|t)$ are the Fourier transforms of fundamental solutions to the wave equation. Moreover, we have

$$\begin{aligned}\hat{\varsigma}_{42}^{l}-\hat{\varsigma}_{44}^{l} = {} & \eta n_{*}c_{*}^{2}\mathrm{e}^{-\kappa|\xi|^{2}t}\Big\{\hat{\omega}\cos(\sigma(|\xi|^{2})|\xi|t)+\\ & \hat{\omega}_{t}\frac{\sin(\sigma(|\xi|^{2})|\xi|t)}{c_{*}\sigma(|\xi|^{2})|\xi|t}\sigma(|\xi|^{2})t\Big\}\xi\xi^{\mathrm{T}}+\\ & \mathrm{e}^{-\frac{1}{\eta n_{*}}t}\left(O(1)|\xi|^{2}t\boldsymbol{I}_{3}+O(1)t\xi\xi^{\mathrm{T}}\right),\end{aligned}\tag{25}$$

and the other terms can be treated in the similar argument.

In order to estimate the inverse Fourier transform of $\hat{\varsigma}_{jk}^{l}(j,k=1,2,3,4)$ in the low frequency part $|\xi|\leqslant r$, we first consider

$$J_{1}^{\alpha}(x,t):=\int_{|\xi|\leqslant r}(\mathrm{i}\xi)^{\alpha}\mathrm{e}^{-c_{1}|\xi|^{2}t}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi\tag{26}$$

for a positive constant $c_{1}>0$. For each $x\in\mathbf{R}^{3}$, there exists an orthogonal mapping such that $x\mapsto(|x|,0,0)^{\mathrm{T}}$ and $\xi\mapsto\zeta=(\zeta_{1},\zeta_{2},\zeta_{3})^{\mathrm{T}}$, then (26) can be estimated by

$$\begin{aligned}|J_{1}^{\alpha}| \leqslant {} & C\int_{|\zeta|\leqslant r}\left|(1+t)^{-\frac{\alpha}{2}}\mathrm{e}^{-\frac{1}{2}c_{1}|\zeta|^{2}(1+t)}\mathrm{e}^{\mathrm{i}|x|\zeta_{1}}\right|\mathrm{d}\zeta\leqslant\\ & C(1+t)^{-\frac{\alpha}{2}}\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\left|\mathrm{e}^{-\frac{1}{2}c_{1}|\zeta|^{2}(1+t)}\mathrm{e}^{\mathrm{i}|x|\zeta_{1}}\right|\mathrm{d}\zeta_{1}\mathrm{d}\zeta_{2}\mathrm{d}\zeta_{3}+\\ & C(1+t)^{-\frac{\alpha}{2}}\int_{\frac{r}{\sqrt{3}}\leqslant|\zeta|\leqslant r}\left|\mathrm{e}^{-\frac{1}{2}c_{1}|\zeta|^{2}(1+t)}\mathrm{e}^{\mathrm{i}|x|\zeta_{1}}\right|\mathrm{d}\zeta.\end{aligned}\tag{27}$$

Since $|\zeta|\geqslant\dfrac{r}{\sqrt{3}}$, the second term of (27) is bounded by $C\mathrm{e}^{-\frac{1+t}{C}}$, and then by $C\mathrm{e}^{-\frac{1+t}{C}}\mathrm{e}^{-\frac{|x|^{2}}{C(1+t)}}$ inside the finite Mach region $|x|\leqslant 2Mc_{*}(1+t)$. For the first term of (27), we shift the integration contour of ζ_{1} in the complex plane by choosing the contour as $\Gamma=\Gamma_{1}\cup\Gamma_{2}\cup\Gamma_{3}\cup\Gamma_{4}$, where

$$\begin{aligned}\Gamma_{1} &= \left\{z:\mathrm{Im}z=0,|\mathrm{Re}z|\leqslant\frac{r}{\sqrt{3}}\right\},\\ \Gamma_{2} &= \left\{z:\mathrm{Re}z=-\frac{r}{\sqrt{3}},0\leqslant\mathrm{Im}z\leqslant\frac{r}{\sqrt{3}}\frac{|x|}{(1+t)}\right\},\\ \Gamma_{3} &= \left\{z:\mathrm{Im}z=\frac{r}{\sqrt{3}}\frac{|x|}{(1+t)},|\mathrm{Re}z|\leqslant\frac{r}{\sqrt{3}}\right\},\\ \Gamma_{4} &= \left\{z:\mathrm{Re}z=\frac{r}{\sqrt{3}},0\leqslant\mathrm{Im}z\leqslant\frac{r}{\sqrt{3}}\frac{|x|}{(1+t)}\right\}.\end{aligned}$$

Denote $\zeta_{11}=\mathrm{Re}\zeta_{1}$ and $\zeta_{12}=\mathrm{Im}\zeta_{1}$, one can easily check that on the path Γ_{3}, it holds

$$\begin{aligned}&\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{\Gamma_{3}}\left|\mathrm{e}^{-\frac{1}{2}c_{1}|\zeta|^{2}(1+t)}\mathrm{e}^{\mathrm{i}|x|\zeta_{1}}\right|\mathrm{d}\zeta_{1}\mathrm{d}\zeta_{2}\mathrm{d}\zeta_{3}\leqslant\\ &\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\mathrm{e}^{-\frac{1}{2}c_{1}(\zeta_{2}^{2}+\zeta_{3}^{2})t}\mathrm{e}^{-\frac{r|x|^{2}}{\sqrt{3}(1+t)}+\frac{r^{2}c_{1}|x|^{2}}{6(1+t)}}\mathrm{d}\zeta_{2}\mathrm{d}\zeta_{3}\times\\ &\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\mathrm{e}^{-\frac{1}{2}c_{1}\zeta_{11}^{2}}\mathrm{d}\zeta_{11}\leqslant C(1+t)^{-\frac{3}{2}}\mathrm{e}^{-\frac{|x|^{2}}{C(1+t)}},\end{aligned}\tag{28}$$

while on the paths Γ_{2} and Γ_{4}, it holds

$$\begin{aligned}&\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{\Gamma_{2},\Gamma_{4}}\left|\mathrm{e}^{-\frac{1}{2}c_{1}|\zeta|^{2}(1+t)}\mathrm{e}^{\mathrm{i}|x|\zeta_{1}}\right|\mathrm{d}\zeta_{1}\mathrm{d}\zeta_{2}\mathrm{d}\zeta_{3}\leqslant\\ &\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\int_{-\frac{r}{\sqrt{3}}}^{\frac{r}{\sqrt{3}}}\mathrm{e}^{-\frac{1}{2}c_{1}(\zeta_{2}^{2}+\zeta_{3}^{2})t}\mathrm{e}^{-\frac{1}{6}r^{2}c_{1}(1+t)}\mathrm{d}\zeta_{2}\mathrm{d}\zeta_{3}\times\end{aligned}$$

$$\int_0^{\frac{r}{\sqrt{3}}\frac{|x|}{(1+t)}} \mathrm{e}^{-|x|\zeta_{12}+\frac{1}{2}\zeta_{12}^2(1+t)}\mathrm{d}\zeta_{12} \leqslant C\mathrm{e}^{-\frac{1+t}{C}} \leqslant C\mathrm{e}^{-\frac{1+t}{C}}\mathrm{e}^{-\frac{|x|^2}{C(1+t)}}. \tag{29}$$

Combining all the above estimates, together with the properties of the fundamental solutions ω and ω_t to the wave equation (refer to Lemma 4.2 in reference [11]), we are able to conclude

$$\begin{aligned} &\left|J_1^\alpha\right| \leqslant C(1+t)^{-\frac{3+|\alpha|}{2}}\mathrm{e}^{-\frac{|x|^2}{C(1+t)}},\\ &\left|\omega * J_1^\alpha\right| \leqslant C(1+t)^{-\frac{3+|\alpha|}{2}}\mathrm{e}^{-\frac{(|x|-c_*t)^2}{C(1+t)}},\\ &\left|\omega_t * J_1^\alpha\right| \leqslant C(1+t)^{-\frac{4+|\alpha|}{2}}\mathrm{e}^{-\frac{(|x|-c_*t)^2}{C(1+t)}}. \end{aligned} \tag{30}$$

Then, we turn to consider

$$J_2^\alpha(x,t) := \int_{\mathbf{R}^3}(\mathrm{i}\xi)^\alpha(\cos(c_*|\xi|t)-1)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}\mathrm{e}^{-c_1|\xi|^2t}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi \tag{31}$$

for a positive constant $c_1>0$. One can directly obtain that

$$\begin{aligned} \left|J_2^\alpha\right| &= \left|\int_{\mathbf{R}^3}(\mathrm{i}\xi)^\alpha\xi\xi^{\mathrm{T}}\int_0^t\frac{-c_*\sin(c_*|\xi|\tau)}{|\xi|}\mathrm{d}\tau\,\mathrm{e}^{-c_1|\xi|^2t}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi\right| =\\ &\left|c_*^2\partial_x^\alpha\nabla\mathrm{div}\int_0^t\int_{\mathbf{R}^3}\frac{\sin(c_*|\xi|\tau)}{c_*|\xi|}\mathrm{e}^{-c_1|\xi|^2t}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi\mathrm{d}\tau\right| =\\ &\left|c_*^2\partial_x^\alpha\nabla\mathrm{div}\int_0^t\frac{\tau}{4\pi(4\pi c_1t)^{3/2}}\int_{|y|=1}\mathrm{e}^{\frac{|x+c_*\tau y|^2}{4c_1t}}\mathrm{d}S_y\mathrm{d}\tau\right| \leqslant\\ &C\left|\int_0^t\tau\int_{|y|=1}t^{-\frac{5+|\alpha|}{2}}\mathrm{e}^{-\frac{|x+c_*\tau y|^2}{Ct}}\mathrm{d}S_y\mathrm{d}\tau\right| \leqslant\\ &C(1+t)^{-\frac{3+|\alpha|}{2}}\left\{(1+t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(|x|-c_*t)^2}{C(1+t)}}+\mathrm{e}^{-\frac{|x|^2}{C(1+t)}}+\right.\\ &\left.\chi_{\{|x|\leqslant c_*(1+t)\}}B_{\frac{3+|\alpha|}{2}}(|x|,t)\right\}. \end{aligned} \tag{32}$$

Finally, we consider

$$J_3^\alpha(x,t) := \int_{|\xi|\leqslant r}(\mathrm{i}\xi)^\alpha\left(\mathrm{e}^{-c_1|\xi|^2t}-\mathrm{e}^{-c_2|\xi|^2t}\right)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi \tag{33}$$

with positive constants $c_1, c_2 > 0$ and $c_1 \neq c_2$. Obviously, it yields

$$\begin{aligned} \left|J_3^\alpha\right| &= \left|\int_{|\xi|\leqslant r}(\mathrm{i}\xi)^\alpha\int_{c_2t}^{c_1t}\partial_\tau(\mathrm{e}^{-|\xi|^2\tau})\mathrm{d}\tau\frac{\xi\xi^{\mathrm{T}}}{|\xi|^2}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi\right| =\\ &\left|-\int_{c_2t}^{c_1t}\int_{|\xi|\leqslant r}(\mathrm{i}\xi)^\alpha\xi\xi^{\mathrm{T}}\mathrm{e}^{-|\xi|^2\tau}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi\,\mathrm{d}\tau\right| \leqslant\\ &C\int_{c_2t}^{c_1t}(1+\tau)^{-\frac{5+|\alpha|}{2}}\mathrm{e}^{-\frac{|x|^2}{C(1+\tau)}}\mathrm{d}\tau \leqslant C(1+t)^{-\frac{3+|\alpha|}{2}}\mathrm{e}^{-\frac{|x|^2}{C(1+t)}}. \end{aligned} \tag{34}$$

The combination of (30), (32) and (34) give rise to the estimates (20)-(23) of the long wave. Thus, the proof is completed.

Next, we investigate the short wave ς^h corresponding to the high frequency $|\xi| \gg 1$. It can be verified from Lemma 2.1 and the Taylor expansion that

$$\begin{aligned} \mathrm{e}^{\lambda_1t} &= \mathrm{e}^{-t}\left(1+\rho_*t|\xi|^{-2}+r_1(t)O(|\xi|^{-4})\right),\\ \mathrm{e}^{\lambda_2t} &= \mathrm{e}^{-\frac{1}{n_*}|\xi|^2t-\frac{\rho_*}{n_*}t}\left(1-\rho_*t|\xi|^{-2}+r_2(t)O(|\xi|^{-4})\right),\\ \mathrm{e}^{\lambda_3t} &= \mathrm{e}^{-t}\left(1-\frac{\eta\rho_*}{c_*^2-\eta}t|\xi|^{-2}+r_3(t)O(|\xi|^{-4})\right),\\ \mathrm{e}^{\lambda_4t} &= \mathrm{e}^{-\frac{c_*^2}{\eta}t}\left(1+\frac{c_*^4(1-c_*^2n_*)}{\eta^2(c_*^2-\eta)}t|\xi|^{-2}+r_4(t)O(|\xi|^{-4})\right),\\ \mathrm{e}^{\lambda_5t} &= \mathrm{e}^{-\frac{1}{n_*}|\xi|^2t-\frac{\eta\rho_*-c_*^2n_*}{\eta n_*}t}\left(1-\frac{\eta^2\rho_*+\eta c_*^2\rho_*-c_*^4n_*}{\eta^2}|\xi|^{-2}+\right.\\ &\left.r_5(t)O(|\xi|^{-4})\right), \end{aligned}$$

where $r_j(t)$ $(j=1,2,\cdots,5)$ are polynomials in t with degree no bigger than 2. Hence, based on the explicit formulas for the elements of $\hat{\varsigma}$, the Fourier transform of the singular leading short wave G_1 is

$$\hat{G}_1 = \mathrm{e}^{-t}\begin{pmatrix} 0 & \mathrm{i}\xi^{\mathrm{T}}-\mathrm{i}\frac{\eta\rho_*}{a_1}p_1(t)\frac{\xi^{\mathrm{T}}}{|\xi|^2} & \frac{c_*^2\rho_*}{a_1n_*}-\frac{\eta c_*^2\rho_*^2}{a_1^2n_*}p_2(t)\frac{1}{|\xi|^2} & -\mathrm{i}\frac{\eta\rho_*}{a_1}\frac{\xi^{\mathrm{T}}}{|\xi|^2}\\ 0 & \boldsymbol{I}_3+\rho_*t\frac{1}{|\xi|^2}\boldsymbol{I}_3-\frac{c_*^2\rho_*}{a_1}p_1(t)\frac{\xi\xi^{\mathrm{T}}}{|\xi|^4} & -\mathrm{i}\frac{c_*^2\rho_*}{a_1n_*}\frac{\xi}{|\xi|^2} & \rho_*\frac{1}{|\xi|^2}\boldsymbol{I}_3-\rho_*\frac{\xi\xi^{\mathrm{T}}}{|\xi|^4}\\ 0 & -\mathrm{i}\frac{\eta n_*}{a_1}\frac{\xi^{\mathrm{T}}}{|\xi|^2} & -\frac{\eta c_*^2\rho_*}{a_1^2}\frac{1}{|\xi|^2} & 0\\ 0 & n_*\frac{1}{|\xi|^2}\boldsymbol{I}_3 & 0 & 0 \end{pmatrix}+$$

$$\mathrm{e}^{-\frac{c_*^2}{\eta}t}\begin{pmatrix} 0 & -\mathrm{i}\dfrac{\eta^2\rho_*}{a_1^2}\dfrac{\xi^{\mathrm{T}}}{|\xi|^2} & -\dfrac{\eta\rho_*}{a_1 n_*}-\dfrac{a_2c_*^4\rho_*}{a_1^2\eta n_*}p_3(t)\dfrac{1}{|\xi|^2} & \mathrm{i}\dfrac{\eta\rho_*}{a_1}\dfrac{\xi^{\mathrm{T}}}{|\xi|^2} \\ 0 & -\dfrac{\eta c_*^2\rho_*}{a_1^2}\dfrac{\xi\xi^{\mathrm{T}}}{|\xi|^4} & \mathrm{i}\dfrac{c_*^2\rho_*}{a_1 n_*}\dfrac{\xi}{|\xi|^2} & \dfrac{c_*^2\rho_*}{a_1}\dfrac{\xi\xi^{\mathrm{T}}}{|\xi|^4} \\ 0 & \mathrm{i}\dfrac{\eta n_*}{a_1}\dfrac{\xi^{\mathrm{T}}}{|\xi|^2} & 1+\dfrac{a_2c_*^4}{a_1\eta^2}p_4(t)\dfrac{1}{|\xi|^2} & -\mathrm{i}n_*\dfrac{\xi^{\mathrm{T}}}{|\xi|^2} \\ 0 & \dfrac{c_*^2 n_*}{a_1}\dfrac{\xi\xi^{\mathrm{T}}}{|\xi|^4} & -\mathrm{i}\dfrac{c_*^2}{\eta}\dfrac{\xi}{|\xi|^2} & -\dfrac{c_*^2 n_*}{\eta}\dfrac{\xi\xi^{\mathrm{T}}}{|\xi|^4} \end{pmatrix}, \tag{35}$$

where the constants $a_1 = c_*^2 - \eta$, $a_2 = 1 - c_*^2 n_*$, and the polynomials $p_j(t)$ for $1 \leqslant j \leqslant 4$ are defined as

$$p_1(t) = t - \frac{\eta}{a_1},$$

$$p_2(t) = t - \frac{a_1 + 2a_2\eta}{a_1\eta\rho_*},$$

$$p_3(t) = t + \frac{a_1\eta + 2a_2\eta^2}{a_1 a_2 c_*^2},$$

$$p_4(t) = t + \frac{a_1\eta c_*^2 n_* + a_2\eta^2}{a_1 a_2 c_*^2}.$$

Set

$$\hat{F}(\xi,t) = \hat{G}_1(\xi,t) - \begin{pmatrix} 0 & \mathrm{i}\mathrm{e}^{-t}\xi^{\mathrm{T}} & \dfrac{c_*^2\rho_*}{a_1 n_*}\mathrm{e}^{-t} - \dfrac{\eta\rho_*}{a_1 n_*}\mathrm{e}^{-\frac{c_*^2}{\eta}t} & 0 \\ 0 & \mathrm{e}^{-t}\boldsymbol{I}_3 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\frac{c_*^2}{\eta}t} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

then, from Proposition 4.4 in reference [13], we give the following description of $F(x,t)$ without proof.

Lemma 2.3 Let $F(x,t)$ be defined by (36). Then, for any given positive integer N, it holds

$$F(\cdot,t) \in L^1(\mathbf{R}^3),$$

and

$$|F(x,t)| \leqslant \begin{cases} C\mathrm{e}^{-\frac{t}{C}}|x|^{-2}, & |x| \leqslant R, \\ C\mathrm{e}^{-\frac{t}{C}}|x|^{-N}, & |x| > R. \end{cases} \tag{37}$$

In addition, we have the following space-time pointwise estimates.

Proposition 2.4 Let α be a multi-index with $|\alpha| \geqslant 0$. There exists a constant $C_2 > 0$ such that it holds for the long wave ς^h that

$$|\partial_x^\alpha(\varsigma^h - G_1)(x,t)| \leqslant C_2\mathrm{e}^{-\frac{t+|x|}{C_2}} + C_2 t^{-\frac{3+|\alpha|}{2}}\mathrm{e}^{-\frac{t}{C_2}}\mathrm{e}^{-\frac{|x|^2}{C_2 t}}, \tag{38}$$

where G_1 is defined by (35).

Proof Since $|\xi| \gg 1$, there exist constants $c_3 > 0$ and $c_4 > 0$ such that $\mathrm{e}^{\lambda_1 t}$, $\mathrm{e}^{\lambda_3 t}$ and $\mathrm{e}^{\lambda_4 t}$ are equivalent to $\mathrm{e}^{-c_3 t}$, while $\mathrm{e}^{\lambda_2 t}$ and $\mathrm{e}^{\lambda_5 t}$ are equivalent to $\mathrm{e}^{-\frac{1}{2n_*}|\xi|^2 t - c_4 t}$. We thus have

$$\begin{aligned} \hat{\varsigma}^h - \hat{G}_1 = {} & \mathrm{e}^{-c_3 t}\begin{pmatrix} 0 & \mathrm{i}O(|\xi|^{-4})\xi^{\mathrm{T}} & O(|\xi|^{-4}) & \mathrm{i}O(|\xi|^{-4})\xi^{\mathrm{T}} \\ 0 & O(|\xi|^{-4}) + O(|\xi|^{-6})\xi\xi^{\mathrm{T}} & \mathrm{i}O(|\xi|^{-4})\xi & O(|\xi|^{-4}) + O(|\xi|^{-6})\xi\xi^{\mathrm{T}} \\ 0 & \mathrm{i}O(|\xi|^{-4})\xi^{\mathrm{T}} & O(|\xi|^{-4}) & \mathrm{i}O(|\xi|^{-4})\xi^{\mathrm{T}} \\ 0 & O(|\xi|^{-4}) + O(|\xi|^{-6})\xi\xi^{\mathrm{T}} & \mathrm{i}O(|\xi|^{-4})\xi & O(|\xi|^{-4}) + O(|\xi|^{-6})\xi\xi^{\mathrm{T}} \end{pmatrix} + \\ & \mathrm{e}^{-\frac{1}{2n_*}|\xi|^2 t - c_4 t}\begin{pmatrix} 0 & \mathrm{i}O(|\xi|^{-6})\xi^{\mathrm{T}} & O(|\xi|^{-4}) & \mathrm{i}O(|\xi|^{-4})\xi^{\mathrm{T}} \\ 0 & O(|\xi|^{-4}) + O(|\xi|^{-6})\xi\xi^{\mathrm{T}} & \mathrm{i}O(|\xi|^{-4})\xi & O(|\xi|^{-2}) + O(|\xi|^{-4})\xi\xi^{\mathrm{T}} \\ 0 & \mathrm{i}O(|\xi|^{-4})\xi^{\mathrm{T}} & O(|\xi|^{-4}) & \mathrm{i}O(|\xi|^{-2})\xi^{\mathrm{T}} \\ 0 & O(|\xi|^{-2}) + O(|\xi|^{-4})\xi\xi^{\mathrm{T}} & \mathrm{i}O(|\xi|^{-2})\xi & O(1) + O(|\xi|^{-2})\xi\xi^{\mathrm{T}} \end{pmatrix}. \end{aligned} \tag{39}$$

First, we consider

$$K_1(x,t) := \int_{|\xi|\geqslant R} \mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-c_3 t}\frac{(1+\mathrm{i}\xi)}{|\xi|^4}\mathrm{d}\xi, \tag{40}$$

$$\begin{aligned} |K_1| &= \mathrm{e}^{-c_3 t}\left|\int_{|\xi|\geqslant R}\left[\cos(x\cdot\xi)\frac{1}{|\xi|^4}-\sin(x\cdot\xi)\frac{\xi}{|\xi|^4}\right]\mathrm{d}\xi\right| \leqslant \\ &\mathrm{e}^{-c_3 t}\int_0^{2\pi}\int_0^{\pi}\int_R^{+\infty}\left|\cos(|x|\zeta\cos\Theta(\vartheta,\varphi))\frac{1}{\zeta^2}-\sin(|x|\zeta\cos\Theta(\vartheta,\varphi))\frac{1}{\zeta}\right|\mathrm{d}\zeta\mathrm{d}\varphi\mathrm{d}\vartheta \leqslant C\mathrm{e}^{-c_3 t}\leqslant C\mathrm{e}^{-\frac{t+|x|}{C}}. \end{aligned} \tag{41}$$

To estimate the second term on the right hand side of (39), we consider

$$K_2(x,t) := \int_{|\xi|\geqslant R}\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{e}^{-\frac{1}{2n_*}|\xi|^2 t-c_4 t}|\xi|^{-m}\mathrm{d}\xi \tag{42}$$

for any integer $m\geqslant 0$, and it yields

$$|K_2|\leqslant C\mathrm{e}^{-c_4 t}\left|\int_{|\xi|\geqslant R}\mathrm{e}^{-\frac{1}{2n_*}|\xi|^2 t}\mathrm{d}\xi\right|\leqslant Ct^{-\frac{3}{2}}\mathrm{e}^{-\frac{t}{C}}\mathrm{e}^{-\frac{|x|^2}{Ct}}. \tag{43}$$

The combination of (39), (41) and (43) suffice to deduce (38), and we complete the proof.

Then, we study ς^m corresponding to the middle frequency part and give the pointwise behaviors below.

Proposition 2.5 Let α be a multi-index with $|\alpha|\geqslant 0$. There exists a constant $C_3>0$ such that it holds for the middle part ς^m that

$$|\partial_x^\alpha\varsigma^m(x,t)|\leqslant C_3\mathrm{e}^{-\frac{t+|x|}{C_3}}. \tag{44}$$

Proof Since $|\xi|\geqslant r$, there exists a positive constant $C>0$ such that

$$\mathrm{Re}\lambda_j(\xi)\leqslant -C \quad \text{for} \quad j=1,2,\cdots,5, \tag{45}$$

and this, together with the expression of $\hat{\varsigma}(\xi,t)$ and the fact that $|\xi|\leqslant R$, implies

$$\hat{\varsigma}^m(\xi,t)=O(1)\mathrm{e}^{-Ct}. \tag{46}$$

Therefore, inside the finite Mach region, we can obtain (44) and complete the proof.

3 Energy estimates outside the finite Mach region

In this section, we devote to study the Green's function G outside the finite Mach region $|x|\geqslant Mc_*(1+t)$. Noticing the initial singularity of the Green's function at $x=0$, as shown in Section 2, we consider the initial boundary value problem as follows

$$\begin{cases} \partial_t G-\boldsymbol{L}G=0, |x|>Mc_*(1+t), t>0, \\ G(x,0)=0, \\ G(x,t)\big|_{|x|=Mc_*(1+t)}=\varsigma(x,t)\big|_{|x|=Mc_*(1+t)}. \end{cases} \tag{47}$$

We have the following result about the pointwise estimates of the Green's function.

Proposition 3.1 Let α be a multi-index with $|\alpha|\geqslant 0$. There exists a positive constant $M>1$ such that as $|x|\geqslant Mc_*(1+t)$, it holds for the Green's function $G(x,t)$ that

$$|\partial_{\boldsymbol{x}}^\alpha G(x,t)|\leqslant C_4\mathrm{e}^{-\frac{t+|x|}{C_4}}, \tag{48}$$

where $C_4>0$ is a constant independent of the space and time.

Proof According to (5), we rewrite (47) into the following form

$$\begin{cases} \partial_t G_{1j}+\mathrm{div}G_{2j}=0, \\ \partial_t G_{2j}+G_{2j}-\dfrac{\rho_*}{n_*}G_{4j}=0, \\ \partial_t G_{3j}+\mathrm{div}G_{4j}=0, \\ \partial_t G_{4j}+c_0^2\nabla G_{3j}-G_{2j}+\dfrac{\rho_*}{n_*}G_{4j}-\dfrac{1}{n_*}\Delta G_{4j}=0, \\ (G_{1j},G_{2j},G_{3j},G_{4j})(x,0)=(0,0,0,0), \\ (G_{1j},G_{2j},G_{3j},G_{4j})^{\mathrm{T}}\big|_{|x|=Mc_*(1+t)}= \\ \qquad \varsigma(x,t)\big|_{|x|=Mc_*(1+t)}, j=1,2,3,4. \end{cases} \tag{49}$$

For simplicity, we denote the region $\{x\in\mathbf{R}^3:|x|>Mc_*(1+t)\}$ by Ω_t and its boundary $\{x\in\mathbf{R}^3:|x|=Mc_*(1+t)\}$ by $\partial\Omega_t$. Define a weight function $w=\mathrm{e}^{\varepsilon(|x|-Mt)}$ with a small positive constant ε and a large positive constant M, which will be determined later.

First of all, multiplying $\{G_{2j}\cdot(49)_2+c_0^2G_{3j}\cdot(49)_3+G_{4j}\cdot(49)_4\}$ by w and integrating the resulting equations over Ω_t, we apply the integration by parts to get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} w\left(\left|G_{2j}\right|^2+c_0^2\left|G_{3j}\right|^2+\left|G_{4j}\right|^2\right)\mathrm{d}x+\left(\frac{1}{2}\varepsilon M+1\right)\int_{\Omega_t} w\left|G_{2j}\right|^2\mathrm{d}x+\\&\frac{1}{2}\varepsilon M c_0^2\int_{\Omega_t} w\left|G_{3j}\right|^2\mathrm{d}x+\left(\frac{1}{2}\varepsilon M+\frac{\rho_*}{n_*}\right)\int_{\Omega_t} w\left|G_{4j}\right|^2\mathrm{d}x+\frac{1}{n_*}\int_{\Omega_t} w\left|\nabla G_{4j}\right|^2\mathrm{d}x=\\&\left(1+\frac{\rho_*}{n_*}\right)\int_{\Omega_t} wG_{2j}\cdot G_{4j}\mathrm{d}x+\varepsilon c_0^2\int_{\Omega_t} wG_{3j}G_{4j}\cdot\frac{x}{|x|}\mathrm{d}x-\frac{\varepsilon}{n_*}\int_{\Omega_t} wG_{4j}\nabla G_{4j}\cdot\frac{x}{|x|}\mathrm{d}x+\\&\frac{1}{2}Mc_*\int_{\partial\Omega_t} w\left(\left|G_{2j}\right|^2+c_0^2\left|G_{3j}\right|^2+\left|G_{4j}\right|^2\right)\frac{x}{|x|}\cdot\mathrm{d}S_x-c_0^2\int_{\partial\Omega_t} wG_{3j}G_{4j}\cdot\mathrm{d}S_x-\frac{1}{n_*}\int_{\partial\Omega_t} wG_{4j}\nabla G_{4j}\cdot\mathrm{d}S_x.\end{aligned}\tag{50}$$

Then, by the Cauchy-Schwarz inequality, there exists a constant $c_5>0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} w\left(\left|G_{2j}\right|^2+c_0^2\left|G_{3j}\right|^2+\left|G_{4j}\right|^2\right)\mathrm{d}x+\\&c_5\int_{\Omega_t} w\left(\left|G_{2j}\right|^2+c_0^2\left|G_{3j}\right|^2+\left|G_{4j}\right|^2\right)\mathrm{d}x\leqslant\\&C\left|\int_{\partial\Omega_t} w\left|G_{2j}\right|^2\frac{x}{|x|}\cdot\mathrm{d}S_x\right|+C\left|\int_{\partial\Omega_t} w\left|G_{3j}\right|^2\frac{x}{|x|}\cdot\mathrm{d}S_x\right|+\\&C\left|\int_{\partial\Omega_t} w\left|G_{4j}\right|^2\frac{x}{|x|}\cdot\mathrm{d}S_x\right|+\\&C\left|\int_{\partial\Omega_t} wG_{3j}G_{4j}\cdot\mathrm{d}S_x\right|+C\left|\int_{\partial\Omega_t} wG_{4j}\nabla G_{4j}\cdot\mathrm{d}S_x\right|,\end{aligned}\tag{51}$$

where we have chosen suitably large M satisfying

$$M>M_1:=\max\left\{1,\frac{\rho_*-n_*}{\varepsilon n_*},\frac{1+c_0^2 n_*}{n_*}+\frac{n_*-\rho_*}{\varepsilon n_*}\right\}.$$

It follows from Propositions 2.2, 2.4, and 2.5 that for $x\in\partial\Omega_t$ with $M>1$, we have

$$\left|\partial_x^\alpha\left(G_{1j},G_{2j},G_{3j},G_{4j}\right)(x,t)\right|\leqslant C\mathrm{e}^{-\frac{t}{C}},\ |\alpha|\geqslant 0,\tag{52}$$

which, together with (51), the smallness of ε, and the Grönwall's inequality, gives rise to

$$\int_{\Omega_t} w\left(\left|G_{2j}\right|^2+\left|G_{3j}\right|^2+\left|G_{4j}\right|^2\right)\mathrm{d}x\leqslant C\mathrm{e}^{-\frac{t}{C}}.\tag{53}$$

The higher order derivatives of $\left(G_{2j},G_{3j},G_{4j}\right)$ can be estimated in a similar way, here we omit the details and conclude for any $|\alpha|\geqslant 0$ that

$$\int_{\Omega_t} w\left(\left|\partial_x^\alpha G_{2j}\right|^2+\left|\partial_x^\alpha G_{3j}\right|^2+\left|\partial_x^\alpha G_{4j}\right|^2\right)\mathrm{d}x\leqslant C\mathrm{e}^{-\frac{t}{C}}.\tag{54}$$

Next, we take $\int_{\Omega_t} w\partial_x^\alpha G_{1j}\partial_x^\alpha(49)_1\,\mathrm{d}x$ for any $|\alpha|\geqslant 0$ and obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} w\left|\partial_x^\alpha G_{1j}\right|^2\mathrm{d}x+\frac{1}{2}\varepsilon M\int_{\Omega_t} w\left|\partial_x^\alpha G_{1j}\right|^2\mathrm{d}x\leqslant\\&\left|-\int_{\Omega_t} w\partial_x^\alpha G_{1j}\partial_x^\alpha\mathrm{div}G_{2j}\mathrm{d}x\right|+\frac{1}{2}Mc_*\left|\int_{\partial\Omega_t} w\left|\partial_x^\alpha G_{1j}\right|^2\frac{x}{|x|}\mathrm{d}S_x\right|\leqslant\\&\frac{1}{2}\int_{\Omega_t} w\left|\partial_x^\alpha G_{1j}\right|^2\mathrm{d}x+\frac{1}{2}\int_{\Omega_t} w\left|\partial_x^{\alpha+1}G_{2j}\right|^2\mathrm{d}x+\\&\frac{1}{2}Mc_*\left|\int_{\partial\Omega_t} w\left|\partial_x^\alpha G_{1j}\right|^2\frac{x}{|x|}\cdot\mathrm{d}S_x\right|.\end{aligned}\tag{55}$$

Further, by choosing $M>\max\{M_1,\frac{1}{\varepsilon}\}$, we get from (52), (54) and Grönwall's inequality that

$$\int_{\Omega_t} w\left|\partial_x^\alpha G_{1j}\right|^2\mathrm{d}x\leqslant C\mathrm{e}^{-\frac{t}{C}}.\tag{56}$$

It is derived from (54), (56), and the Sobolev embedding theorem that

$$\begin{aligned}&w^{\frac{1}{2}}\left(\left|\partial_x^\alpha G_{1j}\right|+\left|\partial_x^\alpha G_{2j}\right|+\left|\partial_x^\alpha G_{3j}\right|+\left|\partial_x^\alpha G_{4j}\right|\right)\leqslant\\&C\mathrm{e}^{-\frac{t}{C}},\ |\alpha|\geqslant 0,\end{aligned}\tag{57}$$

which, together with the smallness of $\varepsilon>0$, leads to (48). Thus, we complete the proof.

The proof of Theorem 1 The space-time pointwise estimates of the Green's function in Theorem 1 can be derived from the decomposition (11) and (18), the combination of Propositions 2.2, 2.4, 2.5 and 3.1.

4 Cauchy problem for linearized system

In this section, we investigate the space-time pointwise behaviors of global solutions to the Cauchy problem (3). Let $U:=(\tilde{\rho},\phi,\tilde{n},\psi)$ be the global solution to system (3) with the initial data $U_0:=(\tilde{\rho}_0,\phi_0,\tilde{n}_0,\psi_0)$, then it holds

$$U(x,t)=\int_{\mathbf{R}^3} G(x-y,t)U_0(y)\mathrm{d}y.\tag{58}$$

In order to estimate the convolution between the Green's function G and the initial data U_0, we first provide the following lemma.

Lemma 4.1 There exists a generic constant $C>0$ such that

$$\int_{\mathbf{R}^3} \mathrm{e}^{-\frac{(|x-y|-c_*t)^2}{C(1+t)}}(1+|y|^2)^{-\theta}\mathrm{d}y \leqslant CB_{\frac{3}{2}}(|x|-c_*t,t) \text{ for } \theta>\frac{21}{10}, \tag{59}$$

$$\int_{\mathbf{R}^3} \mathrm{e}^{-\frac{|x-y|^2}{C(1+t)}}(1+|y|^2)^{-\theta}\mathrm{d}y \leqslant CB_{\theta}(|x|,t) \text{ for } \theta>\frac{3}{2}, \tag{60}$$

$$\int_{\mathbf{R}^3} B_{\frac{3}{2}}(|x-y|,t)(1+|y|^2)^{-\theta}\mathrm{d}y \leqslant CB_{\frac{3}{2}}(|x|,t) \text{ for } \theta>\frac{3}{2}, \tag{61}$$

where $B_a(|x|,t):=\left(1+\frac{|x|^2}{1+t}\right)^{-a}$ for the constant $a>0$.

Proof It suffices to prove (59), since the estimates (60) and (61) can be carried out in a similar way. For $(|x|-c_*t)^2 \leqslant 4(1+t)$, it is easy to verify

$$\int_{\mathbf{R}^3} \mathrm{e}^{-\frac{(|x-y|-c_*t)^2}{C(1+t)}}(1+|y|^2)^{-\theta}\mathrm{d}y \leqslant C \leqslant CB_{\theta}(|x|-c_*t,t) \text{ for } \theta>\frac{21}{10}. \tag{62}$$

For $|x| \geqslant c_*t+2\sqrt{1+t}$, we divide the integration into two parts:

(1) for $|y| \geqslant \frac{|x|-c_*t}{2}$, it holds

$$\int_{|y|\geqslant\frac{|x|-c_*t}{2}} \mathrm{e}^{-\frac{(|x-y|-c_*t)^2}{C(1+t)}}(1+|y|^2)^{-\theta}\mathrm{d}y \leqslant C(1+(|x|-c_*t)^2)^{-\frac{3}{2}}\int_{\mathbf{R}^3} \mathrm{e}^{-\frac{(|x-y|-c_*t)^2}{C(1+t)}}(1+|y|^2)^{-(\theta-\frac{3}{2})}\mathrm{d}y \leqslant C(1+(|x|-c_*t)^2)^{-\frac{3}{2}}t^{\frac{3}{2}}\left(\int_{\mathbf{R}^3}(1+|y|^2)^{-\frac{5}{2}(\theta-\frac{3}{2})}\mathrm{d}y\right)^{\frac{2}{5}} \leqslant CB_{\frac{3}{2}}(|x|-c_*t,t) \text{ for } \theta>\frac{21}{10}, \tag{63}$$

(2) for $|y| \leqslant \frac{|x|-c_*t}{2}$, i.e., $|x-y|-c_*t \geqslant \frac{|x|-c_*t}{2}$, it holds

$$\int_{|y|\leqslant\frac{|x|-c_*t}{2}} \mathrm{e}^{-\frac{(|x-y|-c_*t)^2}{C(1+t)}}(1+|y|^2)^{-\theta}\mathrm{d}y \leqslant C\mathrm{e}^{-\frac{(|x|-c_*t)^2}{2C(1+t)}}\int_{\mathbf{R}^3} \mathrm{e}^{-\frac{(|x-y|-c_*t)^2}{2C(1+t)}}(1+|y|^2)^{-\theta}\mathrm{d}y \leqslant CB_{\frac{3}{2}}(|x|-c_*t,t). \tag{64}$$

For $|x| \leqslant c_*t-2\sqrt{1+t}$, we consider the two cases of $|y| \geqslant \frac{c_*t-|x|}{2}$ and $|y| \leqslant \frac{c_*t-|x|}{2}$ (i.e., $c_*t-|x-y| \geqslant \frac{c_*t-|x|}{2}$), and the computations are similar to (63) and (64) respectively. Therefore, the proof of Lemma 4.1 is completed.

The proof of Theorem 2 Due to the pointwise description of the Green's function G in Theorem 1, we have

$$U(x,t)=(G-G_S)*U_0(x,t)+G_S*U_0(x,t). \tag{65}$$

It follows from (6)-(8) and Lemma 4.1 that for $j=1,2,3,4$, the first term on the right-hand side of (65) can be estimated as

$$|(G-G_S)_{j1}*\tilde{\rho}_0|=0,$$

$$|(G-G_S)_{j2}*\phi_0| \leqslant \begin{cases} C(1+t)^{-2}B_{\frac{3}{2}}(|x|-c_*t,t), & j:\text{odd},\\ C(1+t)^{-2}B_{\frac{3}{2}}(|x|-c_*t,t)+C(1+t)^{-\frac{3}{2}}B_{\frac{3}{2}}(|x|,t), & j:\text{even}, \end{cases}$$

$$|(G-G_S)_{j3}*\tilde{n}_0| \leqslant C(1+t)^{-2}B_{\frac{3}{2}}(|x|-c_*t,t),$$

$$|(G-G_S)_{j4}*\psi_0| \leqslant \begin{cases} C(1+t)^{-2}B_{\frac{3}{2}}(|x|-c_*t,t), & j:\text{odd},\\ C(1+t)^{-2}B_{\frac{3}{2}}(|x|-c_*t,t)+C(1+t)^{-\frac{3}{2}}B_{\frac{3}{2}}(|x|,t), & j:\text{even}. \end{cases}$$

For the convolution between the singular part G_S and the initial data U_0, we divide it into two parts as

$$G_S*U_0(x,t)=(G_S-F)*U_0(x,t)+F*U_0(x,t), \tag{66}$$

where F is defined by (36). By direct calculations, it holds

$$|((G_S-F)*U_0)_1(x,t)| \leqslant$$

$$C\left|\int_{\mathbf{R}^3}\mathrm{e}^{\mathrm{i}x\cdot\xi}\left(\hat{\tilde{\rho}}_0-\mathrm{i}(1-\mathrm{e}^{-t})\xi^{\mathrm{T}}\cdot\hat{\phi}_0-\frac{\rho_*}{n_*}(1-\frac{c_*^2}{a_1}\mathrm{e}^{-t}+\frac{\eta}{a_1}\mathrm{e}^{-\frac{c_*^2}{\eta}t})\hat{\tilde{n}}_0\right)\mathrm{d}\xi\right|\leqslant$$

$$C\left|(\tilde{\rho}_0-\mathrm{div}\phi_0-\frac{\rho_*}{n_*}\tilde{n}_0)(x)\right|+C\mathrm{e}^{-\frac{t}{C}}\left|(\partial_x\phi_0,\tilde{n}_0)(x)\right|\leqslant$$

$$C(1+|x|^2)^{-\theta}+C\mathrm{e}^{-\frac{t}{C}}(1+|x|^2)^{-\theta},$$

which, combined with (14), yields

$$\left|((G_S-F)*U_0)_1(x,t)\right|\leqslant C\mathrm{e}^{-\frac{t}{C}}(1+|x|^2)^{-\theta}.$$

For $k=2,3,4$, we deduce that

$$\left|((G_S-F)*U_0)_k(x,t)\right|\leqslant C\mathrm{e}^{-\frac{t}{C}}\left|(\phi_0,\tilde{n}_0)(x)\right|\leqslant$$

$$C\mathrm{e}^{-\frac{t}{C}}(1+|x|^2)^{-\theta}.$$

Making use of Lemma 2.3, in a similar argument as the proof of Lemma 4.1, we immediately obtain that

$$\left|F*U_0(x,t)\right|\leqslant C\mathrm{e}^{-\frac{t}{C}}\int_{|x-y|\leqslant R}|x-y|^{-2}(1+|y|^2)^{-\theta}\mathrm{d}y+$$

$$C\mathrm{e}^{-\frac{t}{C}}\int_{|x-y|>R}|x-y|^{-N}(1+|y|^2)^{-\theta}\mathrm{d}y\leqslant$$

$$C\mathrm{e}^{-\frac{t}{C}}(1+|x|^2)^{-\frac{3}{2}}.$$

In summary, we are able to conclude (13) and (15) from all the above estimates, and the proof of Theorem 2 is completed.

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