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一类例外Forelli-Rudin结构的Bergman核函数*

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摘要:一类例外Forelli-Rudin结构是以16维例外Cartan域为底的半Reinhardt域。当这类域的所有参数为正实数时,给出其级数形式的Bergman核函数,并且当其中一个特定的参数为正整数时,得到该域的有限形式的Bergman核函数。

关键词:例外Cartan域;Forelli-Rudin结构;Bergman核函数
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The Bergman kernel function on a certain exceptional Forelli-Rudin construction*

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Abstract: The Forelli-Rudin construction is a type of semi-Reinhardt domain based on the exceptional Cartan domain of dimension 16. The Bergman kernel function on this domain with infinite series form is given when all its parameters are positive real numbers, and the finite sum form of the Bergman kernel function is gotten for a specific parameter being positive integer.

Keywords: exceptional Cartan domain; Forelli-Rudin construction; Bergman kernel function
CLC: O174.56　　**DC**: A

0　引　言

任一有界域都存在唯一的Bergman核函数,而且多复变中许多具体问题的研究都以得到某有界域的Bergman核函数的表达式为基础,如研究Bergman核函数的渐进性质[1-2]、陆启铿猜想[3-4],以及与Bergman度量相关的问题。对Bergman核函数现代理论的研究得到了快速的发展,许多学者在得到不同类型的有界域的Bergman核函数的研究中取得了丰硕的成果。Bergman[5]运用有界Reinhardt域的完备规范正交系和级数求和的技巧,得到了$\mathbb{C}^2$上复椭球域的Bergman核函数;华罗庚[6]运用全纯自同构法给出了4类Cartan域的Bergman核函数;D'angelo[7]运用级数法给出了Thullen域的Bergman核函数

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的表达式;Edigarian和Zwonek[8]利用逆紧全纯映照法得到了对称多圆盘的Bergman核函数;殷慰萍课题组综合运用有界域的全纯自同构、全纯不变量、级数法和膨胀原理等得到了华罗庚域的Bergman核函数[9-11];本课题组给出了几类拟齐性域的Bergman核函数的表达式[12-14]。此外,建立具有一定联系的2个域的Bergman核函数的关系式也是解决Bergman核函数问题的重要方法。如Boas等[3]给出紧缩原理和膨胀原理,得到底空间上的纤维为一维空间与高维空间的2个Hartogs域的Bergman核函数间的关系式;Forelli和Rudin[15]、Ligocka[16]给出一类Hartogs域的Bergman核函数与其底域为单位圆盘,以$1-|z|^2$为权函数的Bergman核函数的关系式;Engliš和Zhang[17]对Forelli等的结论做了进一步推广,当一类Hartogs域底空间上的纤维为任意一个有界对称域时,给出这个Hartogs域的Bergma核函数与其底域的加权Bergman核函数的关系式。最近,在得到Bergman核函数的基础上,关于Bergman核函数的估计问题,Bergman核函数的零点问题,以及关于Bergman度量的d-有界性和消没定理等问题的研究成果可见参考文献[18-22]。

本文将给出以16维例外Cartan域为底的一类Forelli-Rudin结构的Bergman核函数的表达式。这类Forelli-Rudin结构的形式为

$$\Omega=\left\{\xi_1,\xi_2\in\mathbb{C},z\in D:\frac{|\xi_1|^{2\mu_1}}{[N_D(z,z)]^{q_1}}+\frac{|\xi_2|^{2\mu_2}}{[N_D(z,z)]^{q_2}}<1\right\},\mu_k,q_k>0(k=1,2),\quad(1)$$

式中:D表示16维例外Cartan域,$N_D(z,z)$表示D在Jordan三重系下的一般模。关于以第1~4类Cartan域为底的Forelli-Rudin结构的Bergman核函数的结论见参考文献[23-26]。16维例外Cartan域的定义形式和性质不同于前4类Cartan域,从而计算以这类域为底的Forelli-Rudin结构的Bergman核函数的过程更为复杂。本文运用Ω的全纯自同构变换,自全纯不变量和级数求和的技巧,给出了Ω的Bergman核函数表达式的无穷级数形式及有限和形式。而且,当$q_1=q_2=1$,Ω就是以16维例外Cartan域为底的华罗庚域。因此,本文所得到的Ω的Bergman核函数的表达式是在更广泛的一类域上的结果。

1　准备知识

文中Forelli-Rudin结构以16维例外Cartan域为底,本节首先给出应用Jordan三重系来刻画16维例外Cartan域的相关基本概念。

设E是具有单位元$e\neq0$的复八元数系,对(E,n)是复数域$\mathbb{C}$上的8维合成代数,记为$O_{\mathbb{C}}$,称为复Cayley代数。n是E上非奇异的二次型,$n(a)(a\in E)$称为a的Cayley范数。设$a,b\in E$,用$(:)$表示与n相关的双线性形式

$$(a:b)=n(a+b)-n(a)-n(b)。$$

E中的元素a的Cayley共轭定义为$\tilde{a}=(a:e)e-a$。

$H_3(O_{\mathbb{C}})$表示由表值属于$O_{\mathbb{C}}$的3×3矩阵所组成的$\mathbb{C}$-向量空间。$H_3(O_{\mathbb{C}})$中的元素$\boldsymbol{A}$的具体形式为

$$\boldsymbol{A}=\begin{pmatrix}\alpha_1&w_3&\tilde{w}_2\\\tilde{w}_3&\alpha_2&w_1\\w_2&\tilde{w}_1&\alpha_3\end{pmatrix},$$

其中$\alpha_1,\alpha_2,\alpha_3\in\mathbb{C},w_1,w_2,w_3\in O_{\mathbb{C}}$。可设$\boldsymbol{A}=\sum_{j=1}^{3}\alpha_j\boldsymbol{e}_j+\sum_{j=1}^{3}\boldsymbol{F}_j(w_j)$,这里

$$\boldsymbol{e}_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\boldsymbol{e}_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},$$

$$\boldsymbol{e}_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\boldsymbol{F}_1(w_1)=\begin{pmatrix}0&0&0\\0&0&w_1\\0&\tilde{w}_1&0\end{pmatrix},$$

$$\boldsymbol{F}_2(w_2)=\begin{pmatrix}0&0&\tilde{w}_2\\0&0&0\\w_2&0&0\end{pmatrix},\boldsymbol{F}_3(w_3)=\begin{pmatrix}0&w_3&0\\\tilde{w}_3&0&0\\0&0&0\end{pmatrix}。$$

定义$\boldsymbol{A}$的伴随$\boldsymbol{A}^{\#}$为

$$\boldsymbol{A}^{\#}=\sum_{i=1}^{3}(\alpha_j\alpha_k-n(w_i))\boldsymbol{e}_i+\sum_{i=1}^{3}\tilde{\boldsymbol{F}}_i(w_jw_k-\alpha_i\tilde{w}_i),$$

其中j,k满足(i,j,k)是$(1,2,3)$的偶置换,$\tilde{\boldsymbol{F}}_i(c)$表示$\boldsymbol{F}_i(\tilde{c})$。另设

$$\boldsymbol{B}=\sum_{j=1}^{3}\beta_j\boldsymbol{e}_j+\sum_{j=1}^{3}\boldsymbol{F}_j(v_j),$$

定义双线性形式

$$(\boldsymbol{A}:\boldsymbol{B})=\sum_{j=1}^{3}\alpha_j\beta_j+\sum_{j=1}^{3}(w_j:v_j),$$

并且定义对称双线性映射

$$\boldsymbol{A}\times\boldsymbol{B}:=(\boldsymbol{A}+\boldsymbol{B})^{\#}-\boldsymbol{A}^{\#}-\boldsymbol{B}^{\#},\boldsymbol{A}\times\boldsymbol{A}=2\boldsymbol{A}^{\#}$$

为$H_3(O_{\mathbb{C}})$上的Freudenthal积。设$\bar{\boldsymbol{A}}=\sum_{j=1}^{3}\bar{\alpha}_j\boldsymbol{e}_j+$

$\sum_{j=1}^{3} \boldsymbol{F}_j(\bar{w}_j)$表示$\boldsymbol{A}$的复共轭,$(\boldsymbol{A}|\boldsymbol{B}) = (\boldsymbol{A}:\bar{\boldsymbol{B}})$定义为$H_3(O_{\mathbb{C}})$的Hermite标量积。在$H_3(O_{\mathbb{C}})$上定义Jordan三重积

$$Q(\boldsymbol{A})\boldsymbol{B} = (\boldsymbol{A}|\boldsymbol{B})\boldsymbol{A} - \boldsymbol{A}^{\#} \times \bar{\boldsymbol{B}},$$

$$D(\boldsymbol{A},\boldsymbol{B})\boldsymbol{C} = (\boldsymbol{A}|\boldsymbol{B})\boldsymbol{C} + (\boldsymbol{C}|\boldsymbol{B})\boldsymbol{A} - (\boldsymbol{A} \times \boldsymbol{C}) \times \bar{\boldsymbol{B}}。$$

在此三重积下,$H_3(O_{\mathbb{C}})$是正定的Hermite Jordan三重系。

16维向量空间$W = \{(w_2, w_3)|w_2, w_3 \in O_{\mathbb{C}}\}$是$H_3(O_{\mathbb{C}})$的子空间,其元素的矩阵形式为

$$\begin{pmatrix} 0 & w_3 & \tilde{w}_2 \\ \tilde{w}_3 & 0 & 0 \\ w_2 & 0 & 0 \end{pmatrix},$$

这样16维例外Cartan域可定义为

$$D = \{ w \in W | 1 - (w|w) + (w^{\#}|w^{\#}) > 0, 2 - (w|w) > 0 \},$$

并且定义

$$N_D(z, w) = 1 - (z|w) + (z^{\#}|w^{\#})$$

为D在Jordan三重系下的一般模。

以下给出推导Ω的Bergman核函数所用到的定义和定理。

定义1[27] 设G是$\mathbb{C}^{m+n}$中包含原点的有界域,如果G的全纯自同构群包含如下变换

$$\begin{cases} \eta_j = \mathrm{e}^{\sqrt{-1}\theta_j}\xi_j, \quad 0 \leqslant \theta_j < 2\pi, j = 1, 2, \cdots, m, \\ w_l = \mathrm{e}^{\sqrt{-1}\theta} z_l, \quad 0 \leqslant \theta < 2\pi, l = 1, 2, \cdots, n, \end{cases}$$

则称G是半Reinhardt域。事实上,G对$\xi=(\xi_1, \xi_2, \cdots, \xi_m)$而言是Reinhardt的,对$z = (z_1, z_2, \cdots, z_n)$而言是圆型的。

引理2[27] 设G是$\mathbb{C}^{m+n}$中的半Reinhardt域,则$\{ \xi_1^{j_1}\xi_2^{j_2}\cdots\xi_m^{j_m}P_{kl}^{(j)}(z) \}$是$L^2H(G)$的完备规范正交系。其中,$j_r = 0, 1, \cdots$; $r = 1, 2, \cdots m$; $j = (j_1, j_2, \cdots, j_m)$; $k = 0, 1, 2\cdots$; $l = 0, 1, \cdots, n_k$, $n_k = \dfrac{(n+k-1)!}{(n-1)!\ k!}$; $P_{kl}^{(j)}(z)$是关于$z_1, z_2, \cdots, z_n$的k次齐次多项式。对任何固定的$k, j, P_{k1}^{(j)}(z), P_{k2}^{(j)}(z), \cdots, P_{kn_k}^{(j)}(z)$是线性无关的。

引理3 设Ω是式(1)所定义的以16维例外Cartan域为底的一类Forelli-Rudin结构,则下列变换构成Ω的全纯自同构群,记为$\mathrm{Aut}(\Omega)$:

$$\begin{cases} \eta_k = \xi_k [N_D(z_0, z_0)]^{\frac{q_k}{2\mu_k}} [N_D(z, z_0)]^{-\frac{q_k}{\mu_k}} (k = 1, 2), \\ w = \phi(z), \quad \phi \in \mathrm{Aut}(D), \end{cases}$$

其中$\mathrm{Aut}(D)$表示16维例外Cartan域D的全纯自同构变换群,而且对于$z_0 \in D, \phi \in \mathrm{Aut}(D)$,使得$z_0 = \phi^{-1}(0)$。

证明 显然,这些映照是Ω上的双全纯映照。

因为对于$z_0 \in D, \phi \in \mathrm{Aut}(D)$,使得$\phi(z_0) = 0$,而且由于$D$是极小域,所以对任意$z \in D$,$D$的一般模满足关系式

$$N_D(\phi(z), \phi(z)) = N_D(z, z)\frac{N_D(z_0, z_0)}{|N_D(z, z_0)|^2}。 \quad (2)$$

进一步得到

$$\frac{|\eta_1|^{2\mu_1}}{[N_D(w,w)]^{q_1}} + \frac{|\eta_2|^{2\mu_2}}{[N_D(w,w)]^{q_2}} = \frac{|\xi_1|^{2\mu_1}[N_D(z_0,z_0)]^{q_1}}{|N_D(z,z_0)|^{2q_1}[N_D(w,w)]^{q_1}} + \frac{|\xi_2|^{2\mu_2}[N_D(z_0,z_0)]^{q_2}}{|N_D(z,z_0)|^{2q_2}[N_D(w,w)]^{q_2}} = \frac{|\xi_1|^{2\mu_1}}{|N_D(z,z)|^{q_1}} + \frac{|\xi_2|^{2\mu_2}}{|N_D(z,z)|^{q_2}} < 1。$$

因此,上述变换是Ω的全纯自同构。

引理4[28] 设G_1、G_2是$\mathbb{C}^n$中的有界域,$f: G_1 \to G_2$是双全纯映照,若$K_1(z, \bar{z})$、$K_2(w, \bar{w})$分别是G_1、G_2的Bergman核函数,则

$$K_1(z, \bar{z}) = K_2(w, \bar{w})|\det f'(z)|^2,$$

其中$w = f(z)$。

引理5[11] 设$\xi_1, \xi_2 \in \mathbb{C}$,$\xi = (\xi_1, \xi_2)$,$R > 0$,$j_1$、$j_2$是非负实数,则

$$\int_{|\xi_1|^{2\mu_1} + |\xi_2|^{2\mu_2} < R} |\xi_1|^{2j_1}|\xi_2|^{2j_2} \mathrm{d}V(\xi) = R^{\frac{j_1+1}{\mu_1} + \frac{j_2+1}{\mu_2}} \frac{\pi^2}{\mu_1\mu_2} \cdot \frac{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(1 + \frac{j_1+1}{\mu_1} + \frac{j_2+1}{\mu_2}\right)}。$$

引理6[11] 设D是16维例外Cartan域,$N_D(z, z)$表示D在Jordan三重系下的一般模,$\mathrm{Vol}(D)$表示D的体积,对任意$s \in \mathbb{C}$,$\mathrm{Re}\, s > -1$,有

$$\int_D [N_D(z,z)]^s \mathrm{d}V(z) = \frac{8!\ 11!\ \mathrm{Vol}(D)}{6\prod_{j=1}^{8}(s+j)\prod_{j=4}^{11}(s+j)}。$$

引理7[23] 设$f(t)$是关于t的n次多项式

$$f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0,$$

则对于任意非零实数λ,有$f(t) = \sum_{j=0}^{n} c_j \dfrac{\Gamma(\lambda t + j + 1)}{\Gamma(\lambda t + 1)}$,

其中 $c_j=\dfrac{1}{\Gamma(j+1)}\sum\limits_{k=0}^{j}(-1)^k\binom{j}{k}f\left(\dfrac{-k-1}{\lambda}\right)$, $j=0,1,\cdots,n$。

引理8[26]　对任意实数s、t及正整数k,有

$$\frac{\Gamma(s+t+k)}{\Gamma(s+t)}=\frac{\Gamma(k+1)}{\Gamma(s)\Gamma(t)}\sum_{l=0}^{k}\frac{\Gamma(l+s)\Gamma(t+k-l)}{\Gamma(l+1)\Gamma(k+1-l)}。$$

引理9　对任意$(\xi_1,\xi_2,z)\in\Omega$,设

$$x_k=x_k(\xi_k,z)=|\xi_k|^2[N_D(z,z)]^{-\frac{q_k}{\mu_k}}\ (k=1,2),$$

则x_k在Ω的全纯自同构变换下不变,即$x_k(\xi_k,z)=x_k(\eta_k,w)$,其中$(\eta_1,\eta_2,w)=\psi(\xi_1,\xi_2,z)$, $\psi\in\mathrm{Aut}(\Omega)$。$x_k$称为$\Omega$上的自全纯不变量。

证明　根据引理3及式(2),得到

$$x_k(\eta_k,w)=|\eta_k|^2[N_D(w,w)]^{-\frac{q_k}{\mu_k}}=$$
$$|\xi_k|^2[N_D(z_0,z_0)]^{\frac{q_k}{\mu_k}}|N_D(z,z_0)|^{-\frac{2q_k}{\mu_k}}[N_D(w,w)]^{-\frac{q_k}{\mu_k}}=$$
$$|\xi_k|^2[N_D(z,z)]^{-\frac{q_k}{\mu_k}},(k=1,2)。$$

引理10[29]　对于正实数s,正整数$\dfrac{1}{\mu}$,且$|t|^{\mu}<1$,设$g(t)=\sum\limits_{j=0}^{\infty}\dfrac{\Gamma\left(s+\dfrac{j+1}{\mu}\right)}{\Gamma\left(\dfrac{j+1}{\mu}\right)}t^j$,则

$$g(t)=\frac{\partial}{\partial t}\sum_{\alpha=0}^{\frac{1}{\mu}-1}\frac{\Gamma(s)}{(1-\varpi^{\alpha}t^{\mu})^s},$$

式中:$\omega=\mathrm{e}^{2\sqrt{-1}\pi\mu}$, $t^{\mu}=|t|^{\mu}\mathrm{e}^{\sqrt{-1}\tau\mu}$, $-\pi\leqslant\tau=\arg t<\pi$。

2　Ω的Bergman核函数

以下给出Ω的Bergman核函数表达式的无穷级数形式与有限和形式的主要推导过程。

定理1　设Ω是以16维例外Cartan域为底的一类Forelli-Rudin结构,其具体形式如式(1),若$\mu_k>0,q_k>0(k=1,2)$, 则Ω的Bergman核函数的无穷级数形式为

$$K_{\Omega}((\xi_1,\xi_2,z),(\bar{\xi}_1,\bar{\xi}_2,\bar{z}))=$$
$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}\sum_{r_2=0}^{r_1-s_1}\sum_{s_2=0}^{r_2}U_{r_1}V_{r_2}L_1L_2\Gamma(1+r_2-s_2)\cdot$$
$$\frac{\partial^{s_1+s_2}}{\partial x_1^{s_1}\partial x_2^{s_2}}x_1^{s_1}x_2^{s_2}H(x_1,x_2)[N_D(z,z)]^{-\left(12+\frac{q_1}{2\mu_1}+\frac{q_2}{2\mu_2}\right)},$$

其中

$$C=\frac{6\mu_1\mu_2}{\pi^2 8!\,11!\,\mathrm{Vol}(D)},$$
$$L_k=\frac{\Gamma(r_k+1)}{\Gamma(s_k+1)\Gamma(r_k+1-s_k)}\ (k=1,2),$$
$$U_{r_1}=\frac{1}{\Gamma(r_1+1)}\sum_{l=0}^{r_1}(-1)^l\frac{\Gamma(r_1+1)}{\Gamma(l+1)\Gamma(r_1-l+1)}\cdot$$
$$f_1\left(-\frac{(l+1)q_1}{\mu_1}\right),r_1=0,1,\cdots,16,$$
$$V_{r_2}=\frac{1}{\Gamma(r_2+1)}\sum_{l=0}^{r_2}(-1)^l\frac{\Gamma(r_2+1)}{\Gamma(l+1)\Gamma(r_2-l+1)}\cdot$$
$$f_2\left(-\frac{(l+1)\mu_1q_2}{\mu_2q_1}\right),r_2=0,1,\cdots,r_1-s_1,$$
$$x_k=|\xi_k|^2[N_D(z,z)]^{-\frac{q_k}{\mu_k}}(k=1,2),$$
$$H(x_1,x_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{\Gamma\left(1+\dfrac{j_1+1}{\mu_1}+\dfrac{j_2+1}{\mu_2}\right)}{\Gamma\left(\dfrac{j_1+1}{\mu_1}\right)\Gamma\left(\dfrac{j_2+1}{\mu_2}\right)}x_1^{j_1}x_2^{j_2}。$$

证明　对任意$(\xi_1,\xi_2,z)\in\Omega$,存在$\psi_z\in\mathrm{Aut}(\Omega)$,使得$\psi_z(\xi_1,\xi_2,z)=(\eta_1,\eta_2,0)$,根据引理3和4,以及$\left|\det\left(\dfrac{\partial(\eta_1,\eta_2,w)}{\partial(\xi_1,\xi_2,z)}\right)\right|^2_{z_0=z}=[N_D(z,z)]^{-\left(12+\frac{q_1}{2\mu_1}+\frac{q_2}{2\mu_2}\right)}$,得到

$$K_{\Omega}((\xi_1,\xi_2,z),(\bar{\xi}_1,\bar{\xi}_2,\bar{z}))=$$
$$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))[N_D(z,z)]^{-\left(12+\frac{q_1}{2\mu_1}+\frac{q_2}{2\mu_2}\right)}。\tag{3}$$

下面计算$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))$。

Ω是半Reinhard域,根据引理2,$\{\xi_1^{j_1}\xi_2^{j_2}P_{kl}^{(j)}(z)\}$是$L^2H(\Omega)$的完备规范正交系。其中:$j_1=0,1,\cdots;j_2=0,1,\cdots;j=(j_1,j_2);z=(z_1,z_2,\cdots,z_{16});k=0,1,\cdots;l=1,2,\cdots,Q_k$, $Q_k=\dfrac{(15+k)!}{15!\,k!}$。又由Bergman核函数的定义可知

$$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))=\sum_{j,k,l}|\eta_1^{j_1}\eta_2^{j_2}P_{kl}^{(j)}(0)|^2=$$
$$\sum_{|j|\geqslant 0}|\eta_1|^{2j_1}|\eta_2|^{2j_2}|P_{01}^{(j)}(0)|^2,\tag{4}$$

这里$|j|=j_1+j_2$。

因为$\eta_1^{j_1}\eta_2^{j_2}P_{01}^{(j)}(0)=\eta_1^{j_1}\eta_2^{j_2}P_{01}^{(j)}(z)(|j|=0,1,\cdots)$是$L^2H(\Omega)$的完备规范正交系的元素,所以

$$\int_{\Omega}|\eta_1|^{2j_1}|\eta_2|^{2j_2}|P_{01}^{(j)}(0)|^2\,\mathrm{d}V(\eta_1)\mathrm{d}V(\eta_2)\mathrm{d}V(z)=1。$$

进一步得到

$$|P_{01}^{(j)}(0)|^{-2}=\int_D\int_{\frac{|\eta_1|^{2\mu_1}}{[N_D(z,z)]^{q_1}}+\frac{|\eta_2|^{2\mu_2}}{[N_D(z,z)]^{q_2}}<1}|\eta_1|^{2j_1}|\eta_2|^{2j_2}\cdot \mathrm{d}V(\eta_1)\mathrm{d}V(\eta_2)\mathrm{d}V(z)。$$

令

$$C=\frac{6\mu_1\mu_2}{\pi^2 8!\,11!\,\mathrm{Vol}(D)},\tag{5}$$

$$y_1=\frac{q_1(j_1+1)}{\mu_1}+\frac{q_2(j_2+1)}{\mu_2},\tag{6}$$

由引理5和6,可以得到

$$|P_{01}^{(j)}(0)|^2=C\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}\cdot \prod_{l=1}^{8}(y_1+l)\prod_{l=4}^{11}(y_1+l)。$$

令$f_1(y_1)=\prod_{l=1}^{8}(y_1+l)\prod_{l=4}^{11}(y_1+l)$,并且根据引理7得到

$$f_1(y_1)=\sum_{r_1=0}^{16}U_{r_1}\frac{\Gamma\left(1+r_1+\frac{\mu_1}{q_1}y_1\right)}{\Gamma\left(1+\frac{\mu_1}{q_1}y_1\right)},\tag{7}$$

其中

$$U_{r_1}=\frac{1}{\Gamma(r_1+1)}\sum_{k=0}^{r_1}(-1)^k\binom{r_1}{k}f_1\left(-(k+1)\frac{q_1}{\mu_1}\right),\quad r_1=0,1,\cdots,16。\tag{8}$$

把式(6)代入式(7),得到

$$|P_{01}^{(j)}(0)|^2=C\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}\sum_{r_1=0}^{16}U_{r_1}\cdot \frac{\Gamma\left(1+r_1+1+j_1+\frac{\mu_1q_2}{\mu_2q_1}(j_2+1)\right)}{\Gamma\left(1+1+j_1+\frac{\mu_1q_2}{\mu_2q_1}(j_2+1)\right)}。\tag{9}$$

令

$$y_2=\frac{\mu_1q_2}{\mu_2q_1}(j_2+1),\tag{10}$$

$$L_k=\frac{\Gamma(r_k+1)}{\Gamma(s_k+1)\Gamma(r_k+1-s_k)}(k=1,2),\tag{11}$$

$$M_k=\frac{\Gamma(s_k+j_k+1)}{\Gamma(j_k+1)}(k=1,2),\tag{12}$$

$$f_2(y_2)=\frac{\Gamma(y_2+1+r_1-s_1)}{\Gamma(y_2+1)},\tag{13}$$

把式(11)~(13)代入式(9),且应用引理8,得到

$$|P_{01}^{(j)}(0)|^2=C\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}U_{r_1}L_1M_1f_2(y_2)。\tag{14}$$

因为$f_2(y_2)$是关于y_2的r_1-s_1次多项式,再一次应用引理7,可得

$$f_2(y_2)=\sum_{r_2=0}^{r_1-s_1}V_{r_2}\frac{\Gamma\left(1+r_2+\frac{\mu_2q_1}{\mu_1q_2}y_2\right)}{\Gamma\left(1+\frac{\mu_2q_1}{\mu_1q_2}y_2\right)},\tag{15}$$

其中

$$V_{r_2}=\frac{1}{\Gamma(r_2+1)}\sum_{l=0}^{r_2}(-1)^l\binom{r_2}{l}f_2\left(-\frac{(l+1)\mu_1q_2}{\mu_2q_1}\right),\quad r_2=0,1,\cdots,r_1-s_1。$$

把式(10)代入式(15),得到

$$|P_{01}^{(j)}(0)|^2=C\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}\cdot \sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}U_{r_1}L_1M_1\sum_{r_2=0}^{r_1-s_1}V_{r_2}\frac{\Gamma\left(1+r_2+1+j_2\right)}{\Gamma\left(1+1+j_2\right)}。$$

又根据引理8和式(11)~(12)得到

$$|P_{01}^{(j)}(0)|^2=C\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}\cdot \sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}\sum_{r_2=0}^{r_1-s_1}\sum_{s_2=0}^{r_2}U_{r_1}L_1M_1V_{r_2}L_2M_2\Gamma(1+r_2-s_2)。$$

由引理9,

$$x_k=|\xi_k|^2[N_D(z,z)]^{-\frac{q_k}{\mu_k}}(k=1,2),\tag{16}$$

而且x_k在自同构变换下不变,因此$x_k=|\eta_k|^2(k=1,2)$。

根据上面的计算,得到式(4)可表示为

$$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))=$$

$$C\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}\sum_{r_2=0}^{r_1-s_1}\sum_{s_2=0}^{r_2}U_{r_1}V_{r_2}L_1L_2\cdot$$

$$\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}\Gamma(1+r_2-s_2)M_1x_1^{j_1}M_2x_2^{j_2}。$$

又因为

$$M_1x_1^{j_1}M_2x_2^{j_2}=\frac{\Gamma\left(s_1+j_1+1\right)}{\Gamma\left(j_1+1\right)}\cdot$$

$$\frac{\Gamma\left(s_2+j_2+1\right)}{\Gamma\left(j_2+1\right)}x_1^{j_1}x_2^{j_2}=\frac{\partial^{s_1+s_2}}{\partial x_1^{s_1}\partial x_2^{s_2}}x_1^{j_1+s_1}x_2^{j_2+s_2},$$

令

$$H(x_1,x_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}x_1^{j_1}x_2^{j_2},\text{则}$$

$$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))=$$

$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}\sum_{r_2=0}^{r_1-s_1}\sum_{s_2=0}^{r_2}U_{r_1}V_{r_2}L_1L_2\Gamma(1+r_2-s_2)\cdot$$

$$\frac{\partial^{s_1+s_2}}{\partial x_1^{s_1}\partial x_2^{s_2}}x_1^{s_1}x_2^{s_2}H(x_1,x_2)。$$

因此,Ω的Bergman核函数的无穷级数形式是

$$K_{\Omega}((\xi_1,\xi_2,z),(\bar{\xi}_1,\bar{\xi}_2,\bar{z}))=$$

$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}\sum_{r_2=0}^{r_1-s_1}\sum_{s_2=0}^{r_2}U_{r_1}V_{r_2}L_1L_2\Gamma(1+r_2-s_2)\cdot$$

$$\frac{\partial^{s_1+s_2}}{\partial x_1^{s_1}\partial x_2^{s_2}}x_1^{s_1}x_2^{s_2}H(x_1,x_2)[N_D(z,z)]^{-\left(12+\frac{q_1}{2\mu_1}+\frac{q_2}{2\mu_2}\right)}。$$

定理2　设Ω是以16维例外Cartan域为底的一类Forelli-Rudin结构,其具体形式如式(1),若$\frac{1}{\mu_1}$为正整数,$\mu_2>0$,$q_k>0(k=1,2)$,则Ω的Bergman核函数的有限和形式为

$$K_{\Omega}((\xi_1,\xi_2,z),(\bar{\xi}_1,\bar{\xi}_2,\bar{z}))=$$

$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}\sum_{r_2=0}^{r_1-s_1+1}U_{r_1}W_{r_2}L_1\frac{\partial^{s_1}}{\partial x_1^{s_1}}x_1^{s_1}\frac{\partial}{\partial x_1}\cdot$$

$$\sum_{\alpha=0}^{\frac{1}{\mu_1}-1}\frac{\Gamma(r_2+1)}{(1-t)^{r_2+1}}(1-\varpi^{\alpha}x_1^{\mu_1})^{-\left(1+\frac{1}{\mu_2}\right)}[N_D(z,z)]^{-\left(12+\frac{q_1}{2\mu_1}+\frac{q_2}{2\mu_2}\right)},$$

其中C、U_{r_1}、L_1、$x_k(k=1,2)$的形式分别同式(5)、(8)、(11)和(16)。并且$\varpi=e^{2\pi\sqrt{-1}\mu_1}$,

$$t=x_2(1-\varpi^{\alpha}x_1^{\mu_1})^{-\frac{1}{\mu_2}},$$

$$W_{r_2}=\frac{1}{\Gamma(r_2+1)}\sum_{l=0}^{r_2}(-1)^l\binom{r_2}{l}\hat{f}_2(-l-1),$$

$$r_2=0,1,\cdots,r_1-s_1+1。$$

证明　根据定理1证明中式(3)、(4)和(14),以及$M_1x_1^{j_1}=\frac{\partial^{s_1}}{\partial x_1^{s_1}}x_1^{j_1+s_1}$,得到

$$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))=$$

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)\Gamma\left(\frac{j_2+1}{\mu_2}\right)}\cdot$$

$$\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}U_{r_1}L_1M_1f_2(y_2)x_1^{j_1}x_2^{j_2}=$$

$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}U_{r_1}L_1\frac{\partial^{s_1}}{\partial x_1^{s_1}}x_1^{s_1}\sum_{j_2=0}^{\infty}\frac{f_2(y_2)}{\Gamma\left(\frac{j_2+1}{\mu_2}\right)}x_2^{j_2}\cdot$$

$$\sum_{j_1=0}^{\infty}\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)}x_1^{j_1},\tag{17}$$

其中C、U_{r_1}、L_1、M_1、$f_2(y_2)$、$x_k(k=1,2)$的形式分别如式(5)、(8)、(11)~(13)和式(16)。

当$\frac{1}{\mu_1}$是正整数,根据引理10,令

$$g(x_1)=\sum_{j_1=0}^{\infty}\frac{\Gamma\left(1+\frac{j_1+1}{\mu_1}+\frac{j_2+1}{\mu_2}\right)}{\Gamma\left(\frac{j_1+1}{\mu_1}\right)}x_1^{j_1}=$$

$$\frac{\partial}{\partial x_1}\sum_{\alpha=0}^{\frac{1}{\mu_1}-1}\frac{\Gamma\left(1+\frac{j_2+1}{\mu_2}\right)}{(1-\varpi^{\alpha}x_1^{\mu_1})^{1+\frac{j_2+1}{\mu_2}}},\tag{18}$$

式中:$\varpi=e^{2\pi\sqrt{-1}\mu_1}$,$x_1^{\mu_1}=|x_1|^{\mu_1}e^{\sqrt{-1}\tau\mu_1}$,$-\pi\leqslant\tau=\arg x_1<\pi$。

把式(18)代入式(17)并化简,得到

$$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))=$$
$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}U_{r_1}L_1\frac{\partial^{s_1}}{\partial x_1^{s_1}}x_1^{s_1}\frac{\partial}{\partial x_1}\sum_{\alpha=0}^{\frac{1}{\mu_1}-1}(1-\varpi^{\alpha}x_1^{\mu_1})^{-\left(1+\frac{1}{\mu_2}\right)}\cdot \tag{19}$$
$$\sum_{j_2=0}^{\infty}\frac{j_2+1}{\mu_2}f_2(y_2)[x_2(1-\varpi^{\alpha}x_1^{\mu_1})^{-\frac{1}{\mu_2}}]^{j_2}。$$

在式(19)中,令 $t=x_2(1-\varpi^{\alpha}x_1^{\mu_1})^{-\frac{1}{\mu_2}}$,则 $\tilde{f}_2(t)=\sum_{j_2=0}^{\infty}\frac{j_2+1}{\mu_2}f_2(y_2)t^{j_2}$。并且,令 $\hat{f}_2(j_2)=\frac{j_2+1}{\mu_2}f_2(y_2)$,因为 $\hat{f}_2(j_2)$ 是关于 j_2 的 r_1-s_1+1 次多项式,根据引理7得到

$$\hat{f}_2(j_2)=\sum_{r_2=0}^{r_1-s_1+1}W_{r_2}\frac{\Gamma(1+r_2+j_2)}{\Gamma(1+j_2)},$$

其中

$$W_{r_2}=\frac{1}{\Gamma(r_2+1)}\sum_{l=0}^{r_2}(-1)^l\binom{r_2}{l}\hat{f}_2(-l-1),$$
$$r_2=0,1,\cdots,r_1-s_1+1。$$

又因为 $\sum_{j_2=0}^{\infty}\frac{\Gamma(1+r_2+j_2)}{\Gamma(1+j_2)}t^{j_2}=\frac{\Gamma(r_2+1)}{(1-t)^{r_2+1}}$,所以

$$\tilde{f}_2(t)=\sum_{r_2=0}^{r_1-s_1+1}W_{r_2}\sum_{j_2=0}^{\infty}\frac{\Gamma(j_2+r_2+1)}{\Gamma(j_2+1)}t^{j_2}=$$
$$\sum_{r_2=0}^{r_1-s_1+1}W_{r_2}\frac{\Gamma(r_2+1)}{(1-t)^{r_2+1}}。 \tag{20}$$

最后,把式(20)代入式(19),即得到当 $\frac{1}{\mu_1}$ 是正整数,$\mu_2>0$ 时,

$$K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))=$$
$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}U_{r_1}L_1\frac{\partial^{s_1}}{\partial x_1^{s_1}}x_1^{s_1}\frac{\partial}{\partial x_1}\sum_{\alpha=0}^{\frac{1}{\mu_1}-1}(1-$$
$$\varpi^{\alpha}x_1^{\mu_1})^{-\left(1+\frac{1}{\mu_2}\right)r_1-s_1+1}\sum_{r_2=0}^{}W_{r_2}\frac{\Gamma(r_2+1)}{(1-t)^{r_2+1}}。$$

因此,根据式(3)得到 Ω 的Bergman核函数的有限和形式是

$$K_{\Omega}((\xi_1,\xi_2,z),(\bar{\xi}_1,\bar{\xi}_2,\bar{z}))=$$
$$C\sum_{r_1=0}^{16}\sum_{s_1=0}^{r_1}\sum_{r_2=0}^{r_1-s_1+1}U_{r_1}W_{r_2}L_1\frac{\partial^{s_1}}{\partial x_1^{s_1}}x_1^{s_1}\cdot$$
$$\frac{\partial}{\partial x_1}\sum_{\alpha=0}^{\frac{1}{\mu_1}-1}\frac{\Gamma(r_2+1)}{(1-t)^{r_2+1}}(1-\varpi^{\alpha}x_1^{\mu_1})^{-\left(1+\frac{1}{\mu_2}\right)}$$
$$[N_D(z,z)]^{-\left(12+\frac{q_1}{2\mu_1}+\frac{q_2}{2\mu_2}\right)}。$$

3 结束语

文中所研究的这一类Forelli-Rudin结构 Ω 的底域是16维例外Cartan域,基于该底域是包含原点的齐性域这一特点,把计算 $K_{\Omega}((\xi_1,\xi_2,z),(\bar{\xi}_1,\bar{\xi}_2,\bar{z}))$ 的表达式转化为计算 $K_{\Omega}((\eta_1,\eta_2,0),(\bar{\eta}_1,\bar{\eta}_2,0))$ 的表达式。根据 Ω 是半Reinhardt域及半Reinhardt域完备规范正交系的形式,运用Bergman核函数在全纯自同构下的变换公式,Ω 的自全纯不变量,Γ函数的性质和级数求和的技巧,得到了 Ω 的Bergman核函数的无穷级数形式。并且,当 Ω 的参数满足 $\frac{1}{\mu_1}$ 是正整数,$\mu_2>0$ 时,得到 Ω 的Bergman核函数的有限和形式。在已知 Ω 的Bergman核函数的基础之上,可以进一步研究这一类域的Bergman核函数的性质、陆启铿问题和与Bergman度量相关的系列问题。

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