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几类无理数的无理测度

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摘要：本文使用广义连分数去逼近 e^r，$\tan(r)$，$\tanh(r)$，$r \in \mathbf{Q}^*$，判定他们是无理数，以及计算出其无理性测度为2。

关键词：广义连分数；无理性测度；无理数
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Irrationality measures of several types of irrational numbers

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Abstract：This paper uses generalized continued fractions to approximate e^r，$\tan(r)$，$\tanh(r)$，$r \in \mathbf{Q}^*$，and shows that they are irrational numbers，and their irrationality measures are 2.

Keywords：generalized continued fractions；irrationality measure；irrational number
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0　Introduction

0.1　Bundschuh's result

To measure 'irrationality' of an irrational number ξ，we define irrationality measure of ξ by $\mu = \mu(\xi)$：

$$\mu = \inf\left\{v:\left|\xi - \frac{p}{q}\right| \leqslant q^{-v} \text{holds only for a finite number of rational number } \frac{p}{q}\right\}.$$

To obtain the irrationality measure of e^r，Bundschuh[1] used Hypergeometric series

$${}_1F_1(a;b;x) = \sum_{j=0}^{\infty}\frac{x^j}{j!}\prod_{i=0}^{j=1}\frac{a+i}{b+i}.$$

It satisfies

$${}_1F_1(a;b;x) = e_1^x F_1(b-a;b;-x). \quad (1)$$

He defined

$$P_r(x) = \sum_{j=0}^{r}\binom{2r-j}{r}\frac{(-x)^j}{j!},\ Q_r(x) = P_r(-x),$$

$$R_{2r+1}(x) = \frac{(-1)^r}{2}\sum_{j=2r+1}^{\infty}\binom{j-r-1}{r}\frac{(-x)^j}{j!},$$

$$S_{2r+1}(x) = e^{-x}R_{2r+1}(-x) - R_{2r+1}(x).$$

Using equation（1）with $a = -r, b = 2a$，he got an equality

$$P_r(x) - e^{-x}Q_r(x) = S_{2r+1}(x),$$

where $S_{2r+1}(x) = \frac{(-1)^r x^{2r+1}}{(2r+1)!}$ + higher order. Hence he got the rational relation of $P_r(x)$ and $Q_r(x)$. Consequently，he obtained a rational approximation of e^r. Then he used the rational approximation to show the

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irrationality measure of e^r ($r \in \mathbf{Q}^*$) is 2. Actually, one has the following theorem.

Theorem 1 For each $r \in \mathbf{Q}^*$, e^r is irrational, and the irrationality measure of e^r is 2.

0.2 Generalized continued fraction (GCF)

In order to obtain the irrational measure of a number, it is necessary to give its good rationality approximation. Continued fraction is a tool for good rational approximations of real numbers. In fact, continued fraction can provide the best rational approximation. However, it is difficult to obtain the continued fraction expansion of e^r with $r \in \mathbf{Q}^*$.

Hence we turn to generalized continued fraction. We define GCF by

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}.$$

And we also write

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots.$$

Let P_n and Q_n be the numerator and denominator of the nth convergent, respectively. We say that the general continued fraction is convergent if $\lim\limits_{n\to\infty} \dfrac{P_n}{Q_n}$ exists and is finite. GCF degenerates into continued fraction when $a_1 = a_2 = \cdots = 1$. Therefore, GCF can be regarded as a generalization of continued fraction. But this generalization is not trivial.

GCF was first proposed by Euler[2]. He regarded GCF as a supplementary of infinite series. Lambert used GCF of tan$x$ to prove that π is irrational[3]. Lagrange proposed a method for the solution of differential equations with the help of GCF[4]. Gauss[5] derived from complex-valued hypergeometric functions that are now called Gauss's continued fractions. They can be used to express many elementary functions and some more advanced functions. Apréy[6] proved that $\xi(3)$ is irrational using a fast convergent GCF of $\xi(3)$.

Actually, we can get the GCF of e^r with $r \in \mathbf{Q}^*$ through some less difficult steps. For example, an essential version using recursive formula of linear differential equation can be viewed in reference [4]. For the sake of brevity in this paper, we use a trick using the Bessel function $J_v(x)$ which we will define in Section 2. Then we get the GCF of e^r. The above results show that GCF is not an ordinary generalization of continued fraction since it is difficult to get the continued fraction of e^r.

0.3 Method

Meanwhile, we recall for the nth convergent $\dfrac{P_n}{Q_n}$ in continued fraction that

$$\left|\alpha - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n Q_{n+1}},$$

where α is a real number. Similarly, in GCF, we have

$$\left|\alpha - \frac{P_n}{Q_n}\right| < \frac{a_1 a_2 \cdots a_{n+1}}{Q_n Q_{n+1}}.$$

Actually, a_n in the GCF of e^r is far less than Q_n. Exactly, $\dfrac{a_1 a_2 \cdots a_{n+1}}{Q_n Q_{n+1}}$ is approximately equal to $\dfrac{1}{Q_n Q_{n+1}}$ in the GCF of e^r. The fact makes sure that we can treat the GCF of e^r as continued fraction approximately. Hence it is not hard to get the good rational approximation of e^r. This shows the similarity between GCF and continued fraction.

By using GCF instead of hypergeometric series used by Bundschuh, we reprove Theorem 0.1 due to the fact that the GCF of e^r converges quickly. Incidentally, we get their rationality measures of tan$r$, tanh$r$ with $r \in \mathbf{Q}^*$. Significantly, our fast convergence GCF of e^r, tan$r$, tanh$r$ can directly derive irrationality due to Lemma 1 which means rational number cannot be well approximated by rational number. Actually we have the following theorem.

Theorem 2 For $r \in \mathbf{Q}^*$ we have

(1) tan$r$ is irrational, whose irrationality measure is 2,

(2) tanh$r$ is irrational, whose irrationality measure is 2.

Outline We provide necessary results for irrationality measures in Section 1. We give basic properties of GCF in Section 2. In this section we also give

the GCF of e^r, tan$r$ and tanh$r$. We prove our main theorem in Section 3.

1 Irrationality measures

Lemma 1 Let θ be a real number. If there are infinitely many rational $\frac{p_n}{q_n}$ $(n = 1, 2, \cdots)$ such that $q_n\theta - p_n \neq 0$ and $q_n\theta - p_n \to 0$(when $n \to \infty$). Then θ is irrational.

Proof Our argument comes from reference [7], and we have organized it for the convenience of readers. Assume θ is rational. Let d be the denominator of θ. Then $d\left(q_n\theta - p_n\right) \in \mathbf{Z}$. It means $d\left|q_n\theta - p_n\right| \geqslant 1$, which contradicts $q_n\theta - p_n \to 0$(when $n \to \infty$).

Lemma 2 If there is an integer $q_0 > 0$ such that

$$\left|\xi - \frac{p}{q}\right| > Cq^{-a}.$$

When $q > q_0$, then $\mu(\xi) \leqslant a$.

Proof The proof process can be found in reference [7], and we have organized it for the convenience of readers. If $C \geqslant 1$, then there are only a finite number of rational $\frac{p}{q}$ satisfying

$$\left|\xi - \frac{p}{q}\right| \leqslant q^{-a}.$$

It implies $\mu(\xi) \leqslant a$.

If $C \leqslant 1$, we take an arbitrary ε such that $q_0 > C^{-1/\varepsilon}$. Then there are only a finite number of rational $\frac{p}{q}$ satisfying

$$\left|\xi - \frac{p}{q}\right| \leqslant q^{-(a+\varepsilon)}.$$

It implies $\mu(\xi) \leqslant a + \varepsilon$. Since ε is arbitrary, $\mu(\xi) \leqslant a$.

Lemma 3 If ξ is irrational, then $\mu(\xi) \geqslant 2$.

Proof The proof process can be found in reference [7], and we have organized it for the convenience of readers. Let $Q > 1$ be an integer. Divide $[0, 1]$ into Q subintervals:

$$\left[0, \frac{1}{Q}\right), \left[\frac{1}{Q}, \frac{2}{Q}\right), \cdots, \left[\frac{Q-1}{Q}, 1\right].$$

By the pigeonhole principle, at least two of the $Q + 1$ real numbers $0, \{\xi\}, \{2\xi\}, \cdots, \{(Q-1)\xi\}$ are in the same subinterval. Denote them by a, b. Write $a = r_1\xi + s_1$, $b = r_2\xi + s_2$ with $r_1, r_2 \in \{0, 1, \cdots, Q-1\}$. Then

$$|a - b| = \left|(r_1 - r_2)\xi - (s_2 - s_1)\right| \leqslant \frac{1}{Q}.$$

Assume $r_1 > r_2$. Put $q = r_1 - r_2$ and $q = s_1 - s_2$. Then $0 < q < Q$ and

$$\left|\xi - \frac{p}{q}\right| \leqslant q^{-(a+\varepsilon)}.$$

We claim that there are infinitely many rational $\frac{p}{q}, (p, q) = 1$, and

$$\left|\xi - \frac{p}{q}\right| \leqslant \frac{1}{Qq} < \frac{1}{q^2}. \tag{2}$$

If not, $\xi = \frac{p}{q}$ when $Q \to +\infty$, contradicting that ξ is an irrational number. It means $\mu(\xi) \geqslant 2$ due to (2).

Theorem 3 Given $\alpha \in \mathbf{R}$, assume that there exists a constant $h \geqslant 1$, an increasing function g: $\mathbf{N} \to \mathbf{R}_+^*$ and a rational sequence $\frac{p_n}{q_n}$ of diophantine approximation of α satisfying

$$\forall n \in \mathbf{N},\ q_n p_{n+1} - q_{n+1} p_n \neq 0, \tag{3}$$

$$\forall n \in \mathbf{N},\ \left|q_n\alpha - p_n\right| \ll 1/g(n), \tag{4}$$

$$\forall n \in \mathbf{N},\ \left|q_n\alpha - p_n\right| \ll 1/g(n), \tag{5}$$

$$g(0) = 1,\ \lim_{n \to +\infty} g(n) = +\infty, \tag{6}$$

$$\forall n \in \mathbf{N},\ g(n+1) \ll (g(n))^h. \tag{7}$$

Then we have:

(1) for every rational $\frac{p}{q}$ with $q > 0$,

$$\left|\alpha - \frac{p}{q}\right| \gg \frac{1}{q^\mu},$$

where $\mu = ah^2 + 1$.

(2) the irrationality measure of α satisfies $\mu(\alpha) \leqslant ah^2 + 1$.

Proof (1) Our argument is essentially identical to reference [8]. We give the detail for completeness. Let $(p, q) \in \mathbf{Z} \times \mathbf{N}^*$ be given. Let v be the smallest integer satisfying $q\left|q_v\alpha - p_v\right| \ll \frac{q}{g(v)} < \frac{1}{2}$.

This can be done because of equations (5), (6) and the fact $g(n)$ is increasing. From equation (4) we deduce that $g(v) \ll q^h$. Using equation (4) again we obtain

$$g(v) < g(v+1) \ll q^{h^2}. \tag{8}$$

Now we consider the determinant

$$\Delta_v = \begin{vmatrix} q_v & p_v \\ q_{v+1} & p_{v+1} \end{vmatrix}.$$

From equation (3) we get $\Delta_v \neq 0$ which means the vectors (q_v, p_v) and (q_{v+1}, p_{v+1}) form a basis of $\mathbf{R}^2$. Thus at least one of the two determinants

$$\begin{vmatrix} q_v & p_v \\ q & p \end{vmatrix}, \begin{vmatrix} q_{v+1} & p_{v+1} \\ q & p \end{vmatrix}$$

is not equal to 0. Let $m = v$ or $v+1$ be such that $\delta_m = \begin{vmatrix} q_m & p_m \\ q & p \end{vmatrix} \neq 0$. Obviously $\delta_m \in \mathbf{Z}$, so $|\delta| \geqslant 1$ which means $|pq_m - qp_m| \geqslant 1$. Thus

$$|q(q_m\alpha - p_m) - q_m(q\alpha - p)| \geqslant 1.$$

By the triangle inequality,

$$q|(q_m\alpha - p_m)| + |q_m||(q\alpha - p)| \geqslant 1.$$

By the definitions of v and m, we have

$$|q_m||(q\alpha - p)| > \frac{1}{2}. \tag{9}$$

By equations (5) and (8), we obtain $|q_m| \ll (g(m))^a \ll q^{ah^2}$. By equation (9) we have

$$\left|\alpha - \frac{p}{q}\right| \gg \frac{1}{q^{\mu}},$$

where $\mu = ah^2 + 1$.

(2) It can be deduced from Lemma 2.

2 General continued fractions

Proposition 1 For GCF

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots,$$

we have

(1) $P_n = b_nP_{n-1} + a_nP_{n-2}$, $Q_n = b_nQ_{n-1} + a_nQ_{n-2}$, $(n \geqslant 2)$,

(2) $P_nQ_{n-1} - Q_nP_{n-1} = (-1)^{n+1}a_1a_2\cdots a_n$, $(n \geqslant 1)$,

(3) $\dfrac{P_n}{Q_n} - \dfrac{P_{n-1}}{Q_{n-1}} = (-1)^{n+1}\dfrac{a_1a_2\cdots a_n}{Q_{n-1}Q_n}$,

(4)

$$\frac{P_k}{Q_k} = \frac{P_0}{Q_0} + \frac{a_1}{Q_0Q_1} - \frac{a_1a_2}{Q_1Q_2} + \cdots + (-1)^{k+1}\frac{a_1a_2\cdots a_k}{Q_{k-1}Q_k}.$$

Proof (1) Immediately,

$$\frac{P_0}{Q_0} = \frac{b_0}{1}, \frac{P_1}{Q_1} = \frac{b_0b_1 + a_1}{b_1},$$

$$\frac{P_2}{Q_2} = \frac{b_0b_1b_2 + b_0a_2 + a_1b_2}{b_1b_2 + a_2} = \frac{b_2P_1 + a_2P_0}{b_2Q_1 + a_2Q_0}.$$

Now we use an inductive argument. Assume equation (1) is correct for n. Then

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{\left(b_n + \dfrac{a_{n+1}}{b_{n+1}}\right)P_{n-1} + a_nP_{n-2}}{\left(b_n + \dfrac{a_{n+1}}{b_{n+1}}\right)Q_{n-1} + a_nQ_{n-2}} =$$

$$\frac{b_{n+1}\left(b_nP_{n-1} + a_nP_{n-2}\right) + a_{n+1}P_{n-1}}{b_{n+1}\left(b_nQ_{n-1} + a_nQ_{n-2}\right) + a_{n+1}Q_{n-1}} =$$

$$\frac{b_{n+1}P_n + a_{n+1}P_{n-1}}{b_{n+1}Q_n + a_{n+1}Q_{n-1}}.$$

(2) Immediately, $P_1Q_0 - Q_1P_0 = a_1$. Assume equation (1) is correct for n. Then

$$P_{n+1}Q_n - Q_{n+1}P_n =$$

$$\left(b_{n+1}P_n + a_{n+1}P_{n-1}\right)Q_n - \left(b_{n+1}Q_n + a_{n+1}Q_{n-1}\right)P_n =$$

$$-a_{n+1}\left(P_nQ_{n-1} - Q_nP_{n-1}\right) =$$

$$(-1)^{n+2}a_1a_2\cdots a_{n+1}.$$

(3) Use equation (2).

(4) Add the equalities in equation (3) from $n = 1$ to $n = k$.

Lemma 4 For GCF

$$\alpha = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots,$$

we have

$$\alpha = b_0 + \frac{p_1a_1}{p_1b_1} + \frac{p_1p_2a_2}{p_2b_2} + \cdots + \frac{p_{n-1}p_na_n}{p_nb_n} + \cdots.$$

Proof $\alpha = b_0 + \dfrac{p_1a_1}{p_1b_1} + \dfrac{p_1a_2}{b_2} + \dfrac{a_3}{b_3} + \cdots =$

$$b_0 + \frac{p_1a_1}{p_1b_1} + \frac{p_1p_2a_2}{p_2b_2} + \frac{p_2a_3}{b_3} + \cdots =$$

$$\cdots\cdots$$

$$b_0 + \frac{p_1a_1}{p_1b_1} + \frac{p_1p_2a_2}{p_2b_2} + \cdots + \frac{p_{n-1}p_na_n}{p_nb_n} + \cdots.$$

We define the Bessel function $J_v(x)(x \in \mathbf{C})$ for every $v > -1$ by

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k},$$

where $\Gamma(v)$ is the gamma function.

Proposition 2 For the Bessel function J_v, we have

(1) $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} \sin x$,

(2) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} \cos x$,

(3) $J_{v-1}(x) - J_{v+1}(x) = \left(\frac{2v}{x}\right) J_v(x)$, for $x \neq 0$ and $v > 0$,

(4) $J_v(x) \sim \left(\frac{x}{2}\right)^v \Big/ \Gamma(v+1)$. In particular, $J_v(x) \neq 0 (v > -1)$ for $x \neq 0$.

Proof This can be referred to reference [3], but it does not give full details. We provide a complete proof here.

(1) By the definitions of the Bessel function J_v, observing $\Gamma(v+1) = v\Gamma(v)$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(k+\frac{3}{2}) = (2k+1)!!\sqrt{\pi}/2^{k+1}$ with $k \in \mathbf{N}$, we obtain

$$J_{\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{3}{2})} \left(\frac{x}{2}\right)^{2k} =$$

$$x^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!!(2k+1)!!} x^{2k+1} =$$

$$x^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \sin x.$$

(2) By the definition of the Bessel function J_v, observing $\Gamma(k+\frac{1}{2}) = (2k-1)!!\sqrt{\pi}/2^k$ with $k \in \mathbf{N}$, we obtain

$$J_{-\frac{1}{2}}(x) = \left(\frac{x}{2}\right)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{1}{2})} \left(\frac{x}{2}\right)^{2k} =$$

$$x^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!!(2k-1)!!} x^{2k} =$$

$$x^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \cos x.$$

(3) Observing $\Gamma(v+k+1) = (v+k)\Gamma(v+k)$, we have

$$J_{v-1}(x) = \left(\frac{x}{2}\right)^{v-1} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(v+k)} \left(\frac{x}{2}\right)^{2k} =$$

$$\left(\frac{x}{2}\right)^{v-1} \sum_{k=0}^{+\infty} \frac{(-1)^k (v+k)}{k!\,\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k} =$$

$$v\left(\frac{x}{2}\right)^{v-1} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k} +$$

$$\left(\frac{x}{2}\right)^{v-1} \sum_{k=0}^{+\infty} \frac{(-1)^k k}{k!\,\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k} =$$

$$\frac{2v}{x} J_v(x) + \left(\frac{x}{2}\right)^{v-1} \sum_{k=1}^{+\infty} \frac{(-1)^k}{(k-1)!\,\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k} =$$

$$\frac{2v}{x} J_v(x) - \left(\frac{x}{2}\right)^{v+1} \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\,\Gamma(v+k+2)} \left(\frac{x}{2}\right)^{2k} =$$

$$\frac{2v}{x} J_v(x) - J_{v+1}(x),$$

where $x \neq 0$.

(4) By the definitions of the Bessel function J_v,

$$J_v(x) \Big/ \left(\frac{x}{2}\right)^v = \frac{1}{\Gamma(v+1)} +$$

$$\left(-\frac{1}{1!\,\Gamma(v+2)} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(v+3)} \left(\frac{x}{2}\right)^4 + \cdots\right), \quad (10)$$

denote the series of parentheses in the above equation by A. Noting that $\Gamma(v+1) = v\Gamma(v)$ with $v > 0$ since $\Gamma(v+1) = v\Gamma(v)$ and denoting $\left(\frac{|x|}{2}\right)^2$ by t, we have

$$|A| \leqslant \frac{1}{\Gamma(v+2)} \frac{1}{1!} t + \frac{1}{\Gamma(v+2)} \frac{1}{2!} t^2 +$$

$$\frac{1}{\Gamma(v+2)} \frac{1}{3!} t^3 + \cdots =$$

$$\frac{1}{\Gamma(v+2)} (e^t - 1) = \frac{1}{(v+1)\Gamma(v+1)} (e^t - 1).$$

The final equality is due to

$$\Gamma(v+2) = (v+1)\Gamma(v+1).$$

By equation (10),

$$J_v(x) \sim \left(\frac{x}{2}\right)^v / \Gamma(v+1),$$

Then

$$\frac{J_{v+1}(x)}{J_v(x)} \sim \frac{x}{2(v+1)}. \quad (11)$$

Furthermore, assume $J_v(x) \neq 0$ for some $x \neq 0$ and $v = v_0 - 1 > -1$. By equation(3), we have

$$J_{v_0+1}(x) = \frac{2v_0}{x} J_{v_0}(x).$$

Using equation(3) again, we have

$$J_{v_0+2}(x)=\frac{2\left(v_0+1\right)}{x}J_{v_0+1}(x)-J_{v_0}(x)=\left(\frac{2\left(v_0+1\right)}{x}\frac{2v_0}{x}-1\right)J_{v_0}(x).$$

We can deduce that

$$\frac{J_{v+1}(x)}{J_v(x)}\sim\frac{2v}{x},$$

contradicting with equation (11).

Lemma 5

(1) $\tan x=\frac{x}{1}-\frac{x^2}{3}-\frac{x^2}{5}-\cdots-\frac{x^2}{2n+1}-\cdots$,

(2)For $r=\frac{u}{v}$, $\tan r=\frac{u}{v}-\frac{u^2}{3v}-\frac{u^2}{5v}-\cdots-\frac{u^2}{(2n+1)v}-\cdots$.

Proof (1) The method of proof is attributed to reference [8], but it is only sketched in the place cited. For completeness, we provide a complete proof. Using equations (3) and (4) of Proposition 2, for $x\neq 0$, we have

$$\frac{J_{v-1}(x)}{J_v(x)}=\frac{2v}{x}-\frac{J_{v+1}(x)}{J_v(x)}.$$

Hence

$$\frac{J_v(x)}{J_{v-1}(x)}=\frac{1}{\frac{2v}{x}-\frac{J_{v+1}(x)}{J_v(x)}},$$

then

$$\frac{J_v(x)}{J_{v-1}(x)}=\frac{1}{\frac{2v}{x}}-\frac{1}{\frac{2(v+1)}{x}}-\frac{1}{\frac{2(v+2)}{x}}-\cdots.$$

Using Lemma 4 and putting $p_1=p_2=p_3=\cdots=x$, we obtain

$$\frac{J_v(x)}{J_{v-1}(x)}=\frac{x}{2v}-\frac{x^2}{2v+2}-\frac{x^2}{2v+4}-\cdots.$$

Setting $v=\frac{1}{2}$ in above equation and using equations (1) and (2) of Proposition 2, we get

$$\tan x=\frac{x}{1}-\frac{x^2}{3}-\frac{x^2}{5}-\cdots\frac{x^2}{2n+1}-\cdots.$$

(2) Use Lemma 4 and put $p_1=p_2=p_3=\cdots=v$.

Lemma 6

(1) $\tanh x=\frac{x}{1}+\frac{x^2}{3}+\frac{x^2}{5}+\cdots+\frac{x^2}{2n+1}+\cdots$,

(2) For $r=\frac{u}{v}$, $\tanh r=\frac{u}{v}+\frac{u^2}{3v}+\frac{u^2}{5v}+\cdots+\frac{u^2}{(2n+1)v}+\cdots$.

Proof (1) Replace x by $\frac{x}{\mathrm{i}}$ in Lemma 2.4 and observe $\tanh x=\mathrm{i}\tan\left(\frac{x}{\mathrm{i}}\right)$.

(2) Use Lemma 2.2 and put $p_1=p_2=p_3=\cdots=v$.

Lemma 7

(1) $\mathrm{e}^x=1+\frac{2x}{2-x}+\frac{x^2}{6}+\frac{x^2}{10}+\cdots+\frac{x^2}{4n+2}+\cdots$,

(2) For $r=\frac{u}{v}$, $\mathrm{e}^x=1+\frac{2u}{2v-u}+\frac{u^2}{6v}+\frac{u^2}{10v}+\cdots+\frac{u^2}{(4n+2)v}+\cdots$.

Proof (1) By $\mathrm{e}^r=\frac{1}{1}-\frac{1}{2}-\frac{1}{\mathrm{i}\tan\left(\frac{\mathrm{i}x}{2}\right)}$, Nathan[9] obtained

$$\mathrm{e}^x=1+\frac{-2x}{2+x}+\frac{x^2}{6}+\frac{x^2}{10}+\cdots+\frac{x^2}{4n+2}+\cdots.$$

We give another GCF for e^x through the equation between e^x and $\tanh x$. Recall that $\tanh x=\left(\mathrm{e}^x-\mathrm{e}^{-x}\right)/\left(\mathrm{e}^x+\mathrm{e}^{-x}\right)=\left(\mathrm{e}^{2x}-1\right)/\left(\mathrm{e}^{2x}+1\right)$. Then

$$\mathrm{e}^{2x}=1+\frac{2}{-1+\frac{1}{\tanh x}}. \tag{12}$$

Replacing x by $\frac{x}{2}$ in Lemma 6, we get

$$\tanh\left(\frac{x}{2}\right)=\frac{x/2}{1}+\frac{x^2/4}{3}+\frac{x^2/4}{5}+\cdots+\frac{x^2/4}{2n+1}+\cdots.$$

Putting $p_1=p_2=\cdots=2$ in Lemma 4, we obtain

$$\tanh\left(\frac{x}{2}\right)=\frac{x}{2}+\frac{x^2}{6}+\frac{x^2}{10}+\cdots+\frac{x^2}{4n+2}+\cdots.$$

By equation (12), then

$$e^x = 1 + \cfrac{2}{-1 + \cfrac{1}{\cfrac{x}{2 + \cfrac{x^2}{6 + \cfrac{x^2}{10 + \cdots}}}}} =$$

$$1 + \cfrac{2x}{-x + \cfrac{1}{\cfrac{1}{2 + \cfrac{x^2}{6 + \cfrac{x^2}{10 + \cdots}}}}} =$$

$$1 + \cfrac{2x}{2 - x + \cfrac{x^2}{6 + \cfrac{x^2}{10 + \cdots}}}.$$

(2) Use Lemma 4 and put $p_1 = p_2 = p_3 = \cdots = v$.

3 The proof of main theorem

Lemma 8 For an integer sequence $\{a_n\}$, $a_{n+1} = na_n + sa_{n-1}$ with $a_n \in \mathbf{Z}^*$ and $s \in \mathbf{Z}$, we have the following conclusions.

(1) $\{a_n\}$ is unbounded.

(2) There are infinitely many n such that $|a_{n+1}| > |a_n|$.

(3) There exists n_0 such that $|a_{n+1}| > |a_n|$ when $n > n_0$.

(4) There exists c_1 such that $|a_n| \geqslant c_1(n - |s|)!$.

(5) There exists c_2 such that $|a_n| \leqslant c_2(n + |s|)!$.

(6) For arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, $(n!)^{1-\varepsilon_1} \ll |a_n| \ll (n!)^{1+\varepsilon_2}$.

Proof ((1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6))

(1) If not, then there exists c such that $|a_n| < c$. When $n \geqslant |s|c + c$, we have

$|a_{n+1}| = |na_n + sa_{n-1}| \geqslant n|a_n| - |s|c \geqslant n - |s|c \geqslant c$, contradicting with $|a_n| < c$.

(2) Use (1).

(3) By (2), there exists $n_0 \geqslant |s| + 1$ such that $|a_{n_0}| > |a_{n_0-1}|$. Then

$$|a_{n_0+1}| = |n_0 a_{n_0} + sa_{n_0-1}| \geqslant$$

$$n_0|a_{n_0}| - |s||a_{n_0-1}| =$$

$$(n_0 - |s|)|a_{n_0}| + |s|(|a_{n_0}| - |a_{n_0-1}|) >$$

$$(n_0 - |s|)|a_{n_0}|, \tag{13}$$

and $|a_{n_0+1}| > (n_0 - |s|)|a_{n_0}| \geqslant |a_{n_0}|$.

(4) By equation (13) in (3), there exists n_0 such that $|a_{n+1}| > (n - |s|)|a_n|$ when $n \geqslant n_0$. Then

$$|a_{n+1}| > (n - |s|)|a_n| >$$

$$(n - |s|)(n - |s| - 1)\cdots(n_0 - |s|)|a_{n_0}| =$$

$$c_1(n - |s|)!,$$

where $c_1 = \dfrac{|a_{n_0}|}{(n_0 - |s| - 1)!}$.

(5) From (3), there exists $n_0 > |s|$ such that $|a_n| > |a_{n-1}|$ when $n \geqslant n_0$. Then

$$|a_{n+1}| \leqslant (n + |s|)|a_n| \leqslant$$

$$(n + |s|)(n + |s| - 1)\cdots(n_0 + |s|)|a_{n_0}| =$$

$$c_2(n - |s|)!,$$

where $c_2 = \dfrac{|a_{n_0}|}{(n_0 + |s| - 1)!}$.

(6) Use (4) and (5).

Lemma 9 For an integer sequence $\{a_n\}$, $a_{n+1} = \pm(an + b)a_n + sa_{n-1}$ with $a_n, a \in \mathbf{Z}^*$ and $s, b \in \mathbf{Z}$, we have the following.

(1) There exists n_0 such that $|a_n| > |a_{n-1}|$ when $n > n_0$,

(2) For arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, $(n!)^{1-\varepsilon_1} \ll |a_n| \ll (n!)^{1+\varepsilon_2}$.

Proof The argument is similar to the proof of Lemma 8.

Proposition 3 A meromorphic function $f(x) = \dfrac{x^2}{a+b} + \dfrac{x^2}{2a+b} + \cdots + \dfrac{x^2}{a_n + b}$ for $a \in \mathbf{Z}^*$, converges except for the poles of $f(x)$.

Proof By Proposition 1(1),

$$Q_n = (an + b + a)Q_{n-1} + x^2 Q_{n-2}.$$

And by Lemma 9, for arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,

$$(n!)^{1-\varepsilon_1} \ll |Q_n| \ll (n!)^{1+\varepsilon_2}.$$

Then by Proposition 1(4), for x not belonging to the poles of $f(x)$,

$$\left|f(x)-\frac{P_k}{Q_k}\right|\leqslant\left|(-1)^{k+1}\frac{x^{2k}}{Q_{k-1}Q_k}+(-1)^{k+2}\frac{x^{2k+2}}{Q_kQ_{k+1}}+\cdots\right|\leqslant$$
$$\frac{\left|x^{2k}\right|}{\left|Q_{k-1}Q_k\right|}+\frac{\left|x^{2k+2}\right|}{\left|Q_kQ_{k+1}\right|}+\cdots\ll$$
$$\frac{1}{(k!)^{1-\varepsilon_1}}+\frac{1}{((k+1)!)^{1-\varepsilon_1}}+\cdots\leqslant$$
$$\frac{1}{(k!)^{1-\varepsilon_1}}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)=$$
$$\frac{2}{(k!)^{1-\varepsilon_1}}\to 0.$$

Remark 1 Proposition 3 means the GCFs in Lemmas 5, 6 and 7 converge except for the poles of tan$x$, tanh$x$.

Proof of Theorem 1 (1) Using Lemma 7(2) and Proposition 1(1), we have

$$Q_n=((4n+2)v)Q_{n-1}+u^2Q_{n-2}.$$

And by Lemma 9, for arbitrary $\varepsilon_1>0$ and $\varepsilon_2>0$,

$$(n!)^{1-\varepsilon_1}\ll\left|Q_n\right|\ll(n!)^{1+\varepsilon_2}.$$

By Proposition 1(4) and the fact that the GCF in Lemma 7 converges in the complex plane following form Proposition 3, letting $k\to+\infty$, we obtain

$$e^r=\frac{P_0}{Q_0}+\frac{a_1}{Q_0Q_1}-\frac{a_1a_2}{Q_1Q_2}+\cdots+(-1)^{k+1}\frac{a_1a_2\cdots a_k}{Q_{k-1}Q_k}+\cdots.$$

And using Proposition 2(4) again, we obtain

$$e^r-\frac{P_k}{Q_k}=(-1)^{k+1}\frac{a_1a_2\cdots a_k}{Q_{k-1}Q_k}+\cdots.$$

It means

$$\left|Q_ke^r-P_k\right|\leqslant\frac{\left|a_1a_2\cdots a_{k+1}\right|}{\left|Q_{k+1}\right|}+\frac{\left|a_1a_2\cdots a_{k+2}\right|}{\left|Q_{k+2}\right|}+\cdots=$$
$$\frac{\left|u^{2k+2}\right|}{\left|Q_{k+1}\right|}+\frac{\left|u^{2k+4}\right|}{\left|Q_{k+2}\right|}+\cdots\ll\frac{1}{(k!)^{1-2\varepsilon_1}}.\qquad(14)$$

And by Lemma 1, we have $Q_ke^r-P_k\neq 0$ since the continued fraction of e^r is endless, then e^r is irrational.

Furthermore, observing equation(14), we can choose $g(n)=(n!)^{1-2\varepsilon_1}$, $a=\frac{1+\varepsilon_2}{1-2\varepsilon_1}$, $h=1+\varepsilon$ to make sure that they meet the conditions of Theorem 3, where $\varepsilon,\varepsilon_1,\varepsilon_2$ are arbitrary positive numbers. Using Theorem 3(2), we obtain

$$\mu(e^r)\leqslant ah^2+1=\frac{1+\varepsilon_2}{1-2\varepsilon_1}(1+\varepsilon)^2+1.$$

Since $\varepsilon,\varepsilon_1,\varepsilon_2$ are arbitrary positive numbers, $\mu(e^r)\leqslant 2$. By Lemma 3, $\mu(e^r)=2$.

Remark 2 Theorem 2 can be similarly proven.

Remark 3 By $\tanh x=\mathrm{i}\tan\left(\frac{x}{\mathrm{i}}\right)$ and equation (12), we can conclude that e^r, tan$r$ and tanh$r$ are rationally related. So the irrationality measures of e^r, tan$r$ and tanh$r$ are 2.

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