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量子积分的Ostrowski型不等式

时　统　业*

(中国人民解放军海军指挥学院，江苏 南京　211800)

摘要:针对一阶q导数有界的函数,利用q微积分平均值不等式,建立了q-Ostrowski不等式,加强了已有文献给出的q-Ostrowski不等式,并将结果移植到q^b积分。针对一阶q导数和一阶q^b导数都有界的函数,利用q微积分平均值不等式,建立了同时涉及q积分和q^b积分的Ostrowski型不等式,推广了经典的Ostrowski不等式。针对二阶q可微且二阶q^b可微的函数,利用恒等式,通过引入参数求最值,建立了同时涉及q积分和q^b积分的Ostrowski型不等式。

关键词:Ostrowski型不等式;q积分;q^b积分;量子微积分平均值不等式
中图分类号:O178　　**文献标识码**:A

Ostrowski type inequalities for quantum integration

SHI Tongye*

(Naval Command College of People's Liberation Army, Nanjing Jiangsu　211800)

Abstract: In this paper, q-Ostrowski inequality is established for functions with bounded first order q-derivatives by means of mean value inequality for q-calculus, which strengthens q-Ostrowski inequality given in previous literature, and the result is transferred to q^b-integral. Ostrowski inequality involving both q-integral and q^b-integral is established for functions with bounded first order q-derivatives and q^b-derivatives, and the classical Ostrowski inequality is generalized. For second order q-differentiable and second order q^b-differentiable functions, using the identities and the method of introducing parameter to find the minimum, Ostrowski type inequality involving both q-integral and q^b-integral is established.

Keywords: Ostrowski type inequality; q-integral; q^b-integral; mean value inequality for q-calculus
CLC: O178　　**DC**: A

0　引　言

对于区间$[a,b]$上的可微函数f,如果存在常数M,使得$|f'|\leqslant M$,则有下面经典的Ostrowski不等式

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leqslant\frac{(x-a)^2+(b-x)^2}{2(b-a)}M。$$

Dragomir[1]建立了Lipschitz函数的Ostrowski不等式;Dragomir等[2]给出了带有一个参数的Ostrowski不等式的推广。Ostrowski不等式可以通过建立积分恒等式来证明,利用积分恒等式和Grüss不等式还可以建立Ostrowski-Grüss型不等式[3-5]。许多文献将Ostrowski不等式推广到高阶可微函数的Ostrowski型不等式、加权的Ostrowski型不等式、带有扰动的Ostrowski型不等式,以及各种积分的Ostrowski型不等式[6-11]。更多Ostrowski型不等式的结果可见匡继昌所著的2021年新版《常用不等

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* 通信作者:shtycity@sina.com

式》[12]。本文目的是建立新的量子Ostrowski不等式,通篇假设$0<q<1$。

量子微积分的历史可以追溯到18世纪,欧拉首先在牛顿的无穷级数中引入了参数q;Jackson[13]率先系统地研究q微积分并引入了q定积分的概念;2013年,Tariboon和Ntouyas[14]引入了${}_aD_q$微分算子;2020年,Bermudo等[15]引入q^b导数和q^b积分的概念。许多学者在q-微积分的框架下研究了一些著名的积分不等式,如q-Klamkin-McLenaghan不等式[16]、q-Grueb-Reinboldt不等式[16]、q-Opial不等式[17]、q-Hermite-Hadamard不等式[18]、q^b-Hermite-Hadamard不等式[15]、q-Grüss-Čebyšev不等式[19]和q-Ostrowski不等式[20]等。

定义1[14,19] 设f是$[a,b]$上的连续函数,则f在点$x\in(a,b]$处的q导数定义为

$${}_aD_qf(x)=\frac{f(x)-f\left(qx+(1-q)a\right)}{(1-q)(x-a)}。$$

如果$\lim\limits_{x\to a}{}_aD_qf(x)$存在,则$f$在点$x=a$处的$q$导数定义为

$${}_aD_qf(a)=\lim_{x\to a}{}_aD_qf(x);$$

如果f在$[a,b]$上每个点处的q导数都存在,则称f是$[a,b]$上的q可微函数。

定义2[14] 设f是$[a,b]$上的连续函数,对于任意$x\in(a,b]$,称

$$\frac{{}_aD_qf(x)-{}_aD_qf\left(qx+(1-q)a\right)}{(1-q)(x-a)}$$

为f在点x处的二阶q导数,记作${}_aD_q^2f(x)$,也即${}_aD_q^2f(x)={}_aD_q\left({}_aD_qf(x)\right)$。记$\left\|{}_aD_q^2f\right\|=\sup\limits_{t\in(a,b]}\left|{}_aD_q^2f(t)\right|$。

定义3[14,19] 设f是$[a,b]$上的连续函数,$t\in[a,b]$,则f在$[a,t]$上的q积分定义为

$$\int_a^tf(s)\,{}_ad_qs=(1-q)(t-a)\sum_{n=0}^{\infty}q^nf\left(q^nt+(1-q^n)a\right)。$$

连续函数的q积分一定存在。对于一般的函数f,如果f在$[a,b]$上的q积分存在,则称f在$[a,b]$上是q可积的。Aljinovi等[20]建立了对$x=q^mb+(1-q^m)a$成立的q-Ostrowski型不等式。

定理1[20] 设f是定义在$[a,b]$上的q可积函数,$\left\|{}_aD_qf\right\|=\sup\limits_{t\in(a,b]}\left|{}_aD_qf(t)\right|$存在有限,则有

$$\left|f\left(q^mb+(1-q^m)a\right)-\frac{1}{b-a}\int_a^bf(x)\,{}_ad_qx\right|\leqslant\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}_aD_qf\right\|。$$

Aljinovi等[20]还给出了对任意$x\in[a,b]$都成立的q可积函数的Ostrowski型不等式

$$\left|f(x)-\frac{1}{b-a}\int_a^bf(t)\,{}_ad_qt\right|\leqslant\left(\frac{x-a}{b-a}+\frac{1}{1+q}\right)(b-a)\left\|{}_aD_qf\right\|,$$

式中:f是定义在$[a,b]$上的q可积函数,且f在点$x=a$处连续;$\left\|{}_aD_qf\right\|$存在且有限。

Aljinovi等[20]给出反例说明了经典的微分中值定理对于q可微函数并不成立,但对于形如$q^nx+(1-q^n)a$的2个点一定成立。

定理2[20](q微积分平均值不等式) 设f是$[a,b]$上的任意函数,则对任意$x\in(a,b]$和任意$n\in\mathbf{N}\cup\{0\}$,有

$$\left|f(x)-f\left(q^nx+(1-q^n)a\right)\right|\leqslant(1-q^n)(x-a)\left\|{}_aD_qf\right\|。$$

对任意$m,k\in\mathbf{N}\cup\{0\}$,由定理2直接可得[20]

$$\left|f\left(q^mb+(1-q^m)a\right)-f\left(q^kb+(1-q^k)a\right)\right|\leqslant(b-a)\left|q^m-q^k\right|\left\|{}_aD_qf\right\|。$$

如果f在点$x=a$处连续,取$m\to\infty$得

$$\left|f(a)-f\left(q^kb+(1-q^k)a\right)\right|\leqslant(b-a)\,q^k\left\|{}_aD_qf\right\|。$$

定义4[15] 设f是$[a,b]$上的连续函数,$x\in[a,b]$,则f在点$x\in[a,b)$处的q^b导数定义为

$${}^bD_qf(x)=\frac{f\left(qx+(1-q)b\right)-f(x)}{(1-q)(b-x)}。$$

如果$\lim\limits_{x\to b}{}^bD_qf(x)$存在,则定义${}^bD_qf(b)=\lim\limits_{x\to b}{}^bD_qf(x)$。如果$f$在$[a,b]$上每个点处的$q^b$导数都存在,则称$f$是$[a,b]$上的$q^b$可微函数。

定义5[15] 设f是$[a,b]$上的连续函数,对于任意$x\in[a,b)$,称

$$\frac{{}^bD_qf\left(qx+(1-q)b\right)-{}^bD_qf(x)}{(1-q)(b-x)}$$

为f在点x处的二阶q^b导数,记作${}^bD_q^2f(x)$,也即${}^bD_q^2f(x)={}^bD_q\left({}^bD_qf(x)\right)$,记$\left\|{}^bD_q^2f\right\|=\sup\limits_{t\in[a,b)}\left|{}^bD_q^2f(t)\right|$。

定义6[15] 设f是$[a,b]$上的连续函数,则f在$[a,b]$上的q^b积分定义为

$$\int_a^bf(s)\,{}^bd_qs=(1-q)(b-a)\sum_{n=0}^{\infty}q^nf\left(q^na+(1-q^n)b\right)。$$

连续函数的q^b积分一定存在。对于$[a,b]$上一般的函数f,如果f在$[a,b]$上的q^b积分存在,则称f在$[a,b]$上是q^b可积的。

引理1　设f是$[a,b]$上的任意函数,且$\left\|{}^{b}\mathrm{D}_q f\right\|=\sup\limits_{t\in[a,b)}\left|{}^{b}\mathrm{D}_q f(t)\right|$存在有限,则对任意$k,m\in\mathbf{N}\cup\{0\}$,

$$\left|f\left(q^k a+\left(1-q^k\right)b\right)-f\left(q^m a+\left(1-q^m\right)b\right)\right|\leqslant (b-a)\left|q^k-q^m\right|\left\|{}^{b}\mathrm{D}_q f\right\|。\tag{1}$$

证明　当$k=m$时式(1)显然成立。不妨设$k>m$,则有

$$\left|f\left(q^k a+\left(1-q^k\right)b\right)-f\left(q^m a+\left(1-q^m\right)b\right)\right|=\left|\sum_{n=m}^{k-1}\left[f\left(q^{n+1}a+\left(1-q^{n+1}\right)b\right)-f\left(q^n a+\left(1-q^n\right)b\right)\right]\right|=\left|\sum_{n=m}^{k-1}(1-q)q^n(b-a)\,{}^{b}\mathrm{D}_q f\left(q^n a+\left(1-q^n\right)b\right)\right|\leqslant\sum_{n=m}^{k-1}(1-q)q^n(b-a)\left|{}^{b}\mathrm{D}_q f\left(q^n a+\left(1-q^n\right)b\right)\right|\leqslant\left\|{}^{b}\mathrm{D}_q f\right\|\sum_{n=m}^{k-1}(1-q)q^n(b-a)=(b-a)\left(q^m-q^k\right)\left\|{}^{b}\mathrm{D}_q f\right\|,$$

故式(1)成立。

注1　设f是$[a,b]$上的任意函数,且$\left\|{}^{b}\mathrm{D}_q f\right\|$存在有限,如果$f$在点$x=b$处连续,则对任意$k\in\mathbf{N}\cup\{0\}$,在式(1)中取$m\to\infty$得

$$\left|f\left(q^k a+\left(1-q^k\right)b\right)-f(b)\right|\leqslant(b-a)q^k\left\|{}^{b}\mathrm{D}_q f\right\|。$$

为方便起见,记

$$S_1=\frac{{}_{a}\mathrm{D}_q f(x)-{}_{a}\mathrm{D}_q f(a)}{x-a},S_2=\frac{{}^{b}\mathrm{D}_q f(b)-{}^{b}\mathrm{D}_q f(x)}{b-x}。$$

$$L_1=\int_0^1 t^2\,{}_{a}\mathrm{D}_q^2 f\left(tx+(1-t)a\right)\mathrm{d}_q t,$$

$$L_2=\int_0^1 t^2\,{}^{b}\mathrm{D}_q^2 f\left(tx+(1-t)b\right)\mathrm{d}_q t,$$

$$I_a=\frac{1}{b-a}\int_a^b f(x)\,{}_{a}\mathrm{d}_q x-f\left(q^m b+\left(1-q^m\right)a\right),$$

$$I_b=\frac{1}{b-a}\int_a^b f(x)\,{}^{b}\mathrm{d}_q x-f\left(q^m a+\left(1-q^m\right)b\right),$$

$$J=f(x)-\frac{1}{b-a}\left[\int_a^x f(x)\,{}_{a}\mathrm{d}_q x+\int_x^b f(x)\,{}^{b}\mathrm{d}_q x\right],$$

$$H=\int_a^x f(t)\,{}_{a}\mathrm{d}_q t+\int_x^b f(t)\,{}^{b}\mathrm{d}_q t+\frac{q^2+q-1}{1-q^2}(b-a)f(x)-\frac{q}{1-q^2}\left[(x-a)f\left(qx+(1-q)a\right)+(b-x)f\left(qx+(1-q)b\right)\right]。$$

Ali等[21]通过建立涉及二阶q可微函数的积分恒等式

$$L_1=\frac{1}{(x-a)^3}\left[\frac{q^2+q-1}{q^3(1-q)}(x-a)f(x)-\frac{1}{q^2(1-q)}(x-a)f\left(qx+(1-q)a\right)+\frac{1+q}{q^3}\int_a^x f(t)\,{}_{a}\mathrm{d}_q t\right],\tag{2}$$

$$L_2=\frac{1}{(b-x)^3}\left[\frac{q^2+q-1}{q^3(1-q)}(b-x)f(x)-\frac{1}{q^2(1-q)}(b-x)f\left(qx+(1-q)b\right)+\frac{1+q}{q^3}\int_x^b f(t)\,{}^{b}\mathrm{d}_q t\right],\tag{3}$$

建立了二阶q可微函数的Ostrowski型不等式为

$$\left|{}_{a}^{b}L_q(x)\right|\leqslant\frac{(x-a)^2(b-x)^2(b-a)M}{1+q+q^2},\tag{4}$$

式中:f是$[a,b]$上的二阶q可微函数和二阶q^b可微函数;存在常数M,使得$\left\|{}_{a}\mathrm{D}_q^2 f\right\|\leqslant M,\left\|{}^{b}\mathrm{D}_q^2 f\right\|\leqslant M$,

$${}_{a}^{b}L_q(x)=\frac{(x-a)(b-x)}{(1-q)q^3}\cdot\left[(x-a)qf\left(qx+(1-q)b\right)+(b-x)qf\left(qx+(1-q)a\right)-\left(q^2+q-1\right)(b-a)f(x)\right]-\frac{1+q}{q^3}\left[(b-x)^2\int_a^x f(t)\,{}_{a}\mathrm{d}_q f+(x-a)^2\int_x^b f(t)\,{}^{b}\mathrm{d}_q f\right]。$$

有关q积分不等式和q^b积分不等式的结果还可参阅文献[22-26]。

1　主要结果

定理3　设f是定义在$[a,b]$上的函数,且f在点$x=a$处连续,$\left\|{}_{a}\mathrm{D}_q f\right\|$存在有限且非零,则有

$$-\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}_{a}\mathrm{D}_q f\right\|+\frac{2}{1+q}\left[(1-\bar{u})(1-q\bar{u})+q\bar{v}^2\right](b-a)\left\|{}_{a}\mathrm{D}_q f\right\|\leqslant-\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}_{a}\mathrm{D}_q f\right\|+\frac{2}{1+q}\left\{\left(1-q^{K+1}\right)\left[q^{K+1}-(1+q)\bar{u}+1\right]+q^{L+1}\left[(1+q)\bar{v}-q^{L+1}\right]\right\}(b-a)\left\|{}_{a}\mathrm{D}_q f\right\|\leqslant I_a\leqslant\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}_{a}\mathrm{D}_q f\right\|-$$

$$\frac{2}{1+q}\left\{\left(1-q^{K+1}\right)\left[q^{K+1}-\left(1+q\right)u+1\right]+q^{L+1}\left[\left(1+q\right)v-q^{L+1}\right]\right\}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\leqslant\left(\frac{1+2q^{2m+1}}{1+q}-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|-\frac{2}{1+q}\left[(1-u)(1-qu)+qv^{2}\right](b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,\tag{5}$$

式中:

$$u=\frac{f(b)-f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(1+q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|}{2(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|};$$

$$v=\frac{f\left(q^{m}b+\left(1-q^{m}\right)a\right)-f(a)+q^{m}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|}{2(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|};$$

$$\bar{u}=\frac{f\left(q^{m}b+\left(1-q^{m}\right)a\right)-f(b)+\left(1+q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|}{2(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|};$$

$$\bar{v}=\frac{f(a)-f\left(q^{m}b+\left(1-q^{m}\right)a\right)+q^{m}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|}{2(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|};$$

$$K=\left[\log_{q}u\right],L=\left[\log_{q}v\right]。$$

证明 由定理2,对任意自然数$0\leqslant k\leqslant m$,有

$$f\left(q^{k}b+\left(1-q^{k}\right)a\right)-f(b)\leqslant\left(1-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

$$f\left(q^{k}b+\left(1-q^{k}\right)a\right)-f\left(q^{m}b+\left(1-q^{m}\right)a\right)\leqslant\left(q^{k}-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

故对任意自然数$0\leqslant k\leqslant m$,有

$$f\left(q^{k}b+\left(1-q^{k}\right)a\right)\leqslant\min\left\{f(b)+\left(1-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{k}-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right\}=:\tau,$$

$$\left[f(b)+\left(1-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]-\left[f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{k}-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]=2\left(u-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|。$$

因为$\left|f(b)-f\left(q^{m}b+\left(1-q^{m}\right)a\right)\right|\leqslant\left(1-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|$,故有$u\in\left[q^{m},1\right]$,从而

$$\tau=\begin{cases}f(b)+\left(1-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,\\ \qquad 0\leqslant k\leqslant K;\\ f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{k}-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,\\ \qquad K+1\leqslant k\leqslant m。\end{cases}$$

由定理2,对任意自然数$k\geqslant m+1$,有

$$f\left(q^{k}b+\left(1-q^{k}\right)a\right)-f(a)\leqslant q^{k}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

$$f\left(q^{k}b+\left(1-q^{k}\right)a\right)-f\left(q^{m}b+\left(1-q^{m}\right)a\right)\leqslant\left(q^{m}-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

故对任意自然数$k\geqslant m+1$,有

$$f\left(q^{k}b+\left(1-q^{k}\right)a\right)\leqslant\min\left\{f(a)+q^{k}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{m}-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right\}=:\sigma,$$

$$\left[f(a)+q^{k}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]-\left[f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{m}-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]=2\left(q^{k}-v\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

因为$\left|f\left(q^{m}b+\left(1-q^{m}\right)a\right)-f(a)\right|\leqslant q^{m}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|$,故有$v\in\left[0,q^{m}\right]$,从而

$$\sigma=\begin{cases}f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{m}-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,\\ \qquad m+1\leqslant k\leqslant L;\\ f(a)+q^{k}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,\\ \qquad k\geqslant L+1。\end{cases}$$

$$\int_{a}^{b}f(x)\,{}_{a}\mathrm{d}_{q}x=(1-q)(b-a)\sum_{k=0}^{\infty}q^{k}f\left(q^{k}b+\left(1-q^{k}\right)a\right)=(1-q)(b-a)\left[\sum_{k=0}^{K}q^{k}f\left(q^{k}b+\left(1-q^{k}\right)a\right)+\sum_{k=K+1}^{m}q^{k}f\left(q^{k}b+(1-q^{k})\,a\right)+\sum_{k=m+1}^{L}q^{k}f\left(q^{k}b+(1-q^{k})\,a\right)+\sum_{k=L+1}^{\infty}q^{k}f\left(q^{k}b+\left(1-q^{k}\right)a\right)\right]\leqslant(1-q)(b-a)\left\{\sum_{k=0}^{K}q^{k}\left[f(b)+\left(1-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]+\sum_{k=K+1}^{m}q^{k}\left[f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{k}-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]+\sum_{k=m+1}^{L}q^{k}\left[f\left(q^{m}b+\left(1-q^{m}\right)a\right)+\left(q^{m}-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]+\sum_{k=L+1}^{\infty}q^{k}\left[f(a)+q^{k}(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]\right\},$$

$$I_{a}\leqslant(1-q)\sum_{k=0}^{\infty}q^{k}\left|q^{m}-q^{k}\right|(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|+C+D,\tag{6}$$

式中:

$$C=(1-q)\left\{\sum_{k=0}^{K}q^{k}\left[f(b)+\left(1-q^{k}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]-\sum_{k=0}^{K}q^{k}\left(q^{k}-q^{m}\right)(b-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|-\right.$$

$$f\left(q^m b+\left(1-q^m\right)a\right)\sum_{k=0}^{K}q^k\Bigg\}=$$

$$\frac{2}{1+q}\left(1-q^{K+1}\right)\left[u\left(1+q\right)-1-q^{K+1}\right](b-a)\left\|{}_a\mathrm{D}_q f\right\|;\tag{7}$$

$$D=\left(1-q\right)\Bigg\{\sum_{k=K+1}^{\infty}q^k\left[f\left(a\right)+q^k(b-a)\left\|{}_a\mathrm{D}_q f\right\|\right]-$$

$$\sum_{k=L+1}^{\infty}q^k(q^m-q^k)(b-a)\left\|{}_a\mathrm{D}_q f\right\|-$$

$$f\left(q^m b+\left(1-q^m\right)a\right)\sum_{k=L+1}^{\infty}q^k\Bigg\}=$$

$$\frac{2}{1+q}q^{L+1}\left[q^{L+1}-v\left(1+q\right)\right](b-a)\left\|{}_a\mathrm{D}_q f\right\|。\tag{8}$$

由文献[20]知

$$\sum_{k=0}^{\infty}q^k\left|q^m-q^k\right|=\frac{1}{1-q}\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right),\tag{9}$$

综合式(6)~(9),则式(5)从右边数起第2个不等式得证。

由 $\log_q u-1<K\leqslant\log_q u$,得 $u>q^{K+1}\geqslant qu$,从而有 $\left|q^{K+1}-\frac{1+q}{2}u\right|\leqslant\frac{1-q}{2}u$。同理有

$$\left|q^{L+1}-\frac{1+q}{2}v\right|\leqslant\frac{1-q}{2}v,$$

$$\left(1-q^{K+1}\right)\left[u\left(1+q\right)-1-q^{K+1}\right]=$$

$$\left(q^{K+1}-\frac{1+q}{2}u\right)^2-\left(\frac{1+q}{2}u-1\right)^2\leqslant$$

$$\left(\frac{1-q}{2}u\right)^2-\left(\frac{1+q}{2}u-1\right)^2=(u-1)\left(1-qu\right),$$

$$q^{L+1}\left[q^{L+1}-v\left(1+q\right)\right]=\left(q^{L+1}-\frac{1+q}{2}v\right)^2-$$

$$\left(\frac{1+q}{2}v\right)^2\leqslant\left(\frac{1-q}{2}v\right)^2-\left(\frac{1+q}{2}v\right)^2=-qv^2。$$

故式(5)的右边不等式成立。对$\left(-f\right)$应用已证结果,则式(5)的从左边数起的第1和第2个不等式得证。

推论1　设f是定义在$[a,b]$上的函数,且f在点$x=a$处连续,$\left\|{}_a\mathrm{D}_q f\right\|$存在有限且非零,则有

$$\left|I_a\right|\leqslant\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}_a\mathrm{D}_q f\right\|-$$

$$\frac{\left[\left|f\left(q^m b+\left(1-q^m\right)a\right)-f\left(a\right)\right|-q^m(b-a)\left\|{}_a\mathrm{D}_q f\right\|\right]^2}{2\left(1+q\right)(b-a)\left\|{}_a\mathrm{D}_q f\right\|}。$$

注2　推论1给出定理1的加强。

定理4　设f是定义在$[a,b]$上的函数,且f在点$x=b$处连续,$\left\|{}^b\mathrm{D}_q f\right\|$存在有限且非零,则有

$$-\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|+$$

$$\frac{2}{1+q}\left[(1-\bar{u})(1-q\bar{u})+q\bar{v}^2\right](b-a)\left\|{}^b\mathrm{D}_q f\right\|\leqslant$$

$$-\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|+$$

$$\frac{2}{1+q}\Big\{\left(1-q^{K+1}\right)\left[q^{K+1}-\left(1+q\right)\bar{u}+1\right]+$$

$$q^{L+1}\left[\left(1+q\right)\bar{v}-q^{L+1}\right]\Big\}(b-a)\left\|{}^b\mathrm{D}_q f\right\|\leqslant I_b\leqslant$$

$$\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|-$$

$$\frac{2}{1+q}\Big\{\left(1-q^{K+1}\right)\left[q^{K+1}-\left(1+q\right)u+1\right]+$$

$$q^{L+1}\left[\left(1+q\right)v-q^{L+1}\right]\Big\}(b-a)\left\|{}^b\mathrm{D}_q f\right\|\leqslant$$

$$\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|-$$

$$\frac{2}{1+q}\left[(1-u)(1-qu)+qv^2\right](b-a)\left\|{}^b\mathrm{D}_q f\right\|,$$

其中

$$u=\frac{f\left(a\right)-f\left(q^m a+\left(1-q^m\right)b\right)+\left(1+q^m\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|}{2(b-a)\left\|{}^b\mathrm{D}_q f\right\|};$$

$$v=\frac{f\left(q^m a+\left(1-q^m\right)b\right)-f\left(b\right)+q^m(b-a)\left\|{}^b\mathrm{D}_q f\right\|}{2(b-a)\left\|{}^b\mathrm{D}_q f\right\|};$$

$$\bar{u}=\frac{f\left(a\right)-f\left(q^m a+\left(1-q^m\right)b\right)+\left(1+q^m\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|}{2(b-a)\left\|{}^b\mathrm{D}_q f\right\|};$$

$$\bar{v}=\frac{f\left(b\right)-f\left(q^m a+\left(1-q^m\right)b\right)+q^m(b-a)\left\|{}^b\mathrm{D}_q f\right\|}{2(b-a)\left\|{}^b\mathrm{D}_q f\right\|};$$

$$K=\left[\log_q u\right],L=\left[\log_q v\right]。$$

证明　利用引理1,用类似于定理3的证明方法可证。这里略去证明过程。

推论2　设f是定义在$[a,b]$上的函数,且f在点$x=b$处连续,$\left\|{}^b\mathrm{D}_q f\right\|$存在有限且非零,则有

$$\left|I_b\right|\leqslant\left(\frac{1+2q^{2m+1}}{1+q}-q^m\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|-$$

$$\frac{\left[\left|f\left(q^m a+\left(1-q^m\right)b\right)-f\left(b\right)\right|-q^m(b-a)\left\|{}^b\mathrm{D}_q f\right\|\right]^2}{2\left(1+q\right)(b-a)\left\|{}^b\mathrm{D}_q f\right\|}。$$

定理5 设f是定义在$[a,b]$上的函数,且f在点$x=a$处和点$x=b$处连续,$\left\|{}_{a}\mathrm{D}_{q}f\right\|$和$\left\|{}^{b}\mathrm{D}_{q}f\right\|$存在有限且非零,则对任意$x\in(a,b)$有

$$\frac{q}{(1+q)(b-a)}\left[(2c^{2}-1)(x-a)^{2}\left\|{}_{a}\mathrm{D}_{q}f\right\|+(2d^{2}-1)(b-x)^{2}\left\|{}^{b}\mathrm{D}_{q}f\right\|\right]\leqslant J\leqslant\frac{q}{(1+q)(b-a)}\left[(1-2c_{1}^{2})(x-a)^{2}\left\|{}_{a}\mathrm{D}_{q}f\right\|+(1-2d_{1}^{2})(b-x)^{2}\left\|{}^{b}\mathrm{D}_{q}f\right\|\right],\tag{10}$$

其中

$$c=\frac{f(x)-f(a)+(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|}{2(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|};$$

$$d=\frac{f(x)-f(b)+(b-x)\left\|{}^{b}\mathrm{D}_{q}f\right\|}{2(b-x)\left\|{}^{b}\mathrm{D}_{q}f\right\|};$$

$$c_{1}=\frac{f(x)-f(a)-(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|}{2(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|};$$

$$d_{1}=\frac{f(x)-f(b)-(b-x)\left\|{}^{b}\mathrm{D}_{q}f\right\|}{2(b-x)\left\|{}^{b}\mathrm{D}_{q}f\right\|}。$$

证明 由定理2,对任意自然数k,有

$$f\left(q^{k}x+(1-q^{k})a\right)-f(x)\leqslant(1-q^{k})(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

$$f\left(q^{k}x+(1-q^{k})a\right)-f(a)\leqslant q^{k}(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

故对任意自然数k,有

$$f\left(q^{k}x+(1-q^{k})a\right)\leqslant\min\left\{f(x)+(1-q^{k})(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,\ f(a)+q^{k}(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right\}=:\tau,$$

$$\left[f(x)+(1-q^{k})(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]-\left[f(a)+q^{k}(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]=2(c-q^{k})(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|,$$

由$|f(x)-f(a)|\leqslant(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|$,故$c\in[0,1]$,

$$\tau=\begin{cases}f(x)+(1-q^{k})(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|, & 0\leqslant k\leqslant K;\\ f(a)+q^{k}(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|, & K+1\leqslant k\leqslant m。\end{cases}$$

式中$K=\left[\log_{q}c\right]$。

$$\int_{a}^{x}f(t)\,{}_{a}\mathrm{d}_{q}t=(1-q)(x-a)\left[\sum_{k=0}^{K}q^{k}f\left(q^{k}x+(1-q^{k})a\right)+\sum_{k=K+1}^{\infty}q^{k}f\left(q^{k}x+(1-q^{k})a\right)\right]\leqslant(1-q)(x-a)\cdot\left\{\sum_{k=0}^{K}q^{k}\left[f(x)+(1-q^{k})(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]+\sum_{k=K+1}^{\infty}q^{k}\left[f(a)+q^{k}(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|\right]\right\}=(x-a)\left[(1-q^{K+1})f(x)+\left(1-q^{K+1}-\frac{1-2q^{2(K+1)}}{1+q}\right)\cdot(x-a)\left\|{}_{a}\mathrm{D}_{q}f\right\|+q^{K+1}f(a)\right],$$

$$\int_{a}^{x}f(t)\,{}_{a}\mathrm{d}_{q}t=(1-q)(x-a)\left[\sum_{k=0}^{K}q^{k}f\left(q^{k}x+(1-q^{k})a\right)+\sum_{k=K+1}^{\infty}q^{k}f\left(q^{k}x+(1-q^{k})a\right)\right]\cdot$$

$$(x-a)f(x)-\int_{a}^{x}f(t)\,{}_{a}\mathrm{d}_{q}t\geqslant(x-a)^{2}\left\|{}_{a}\mathrm{D}_{q}f\right\|\left[\frac{1+q}{2}c^{2}-\frac{q}{1+q}-\frac{2}{1+q}\left(q^{K+1}-\frac{1+q}{2}c\right)^{2}\right],$$

由$\log_{q}c-1<K\leqslant\log_{q}c$,得$c>q^{K+1}\geqslant qc$,从而有$\left|q^{K+1}-\frac{1+q}{2}c\right|\leqslant\frac{1-q}{2}c$,于是有

$$(x-a)f(x)-\int_{a}^{x}f(t)\,{}_{a}\mathrm{d}_{q}t\geqslant(x-a)^{2}\left\|{}_{a}\mathrm{D}_{q}f\right\|\left[\frac{1+q}{2}c^{2}-\frac{q}{1+q}-\frac{2}{1+q}\left(\frac{1-q}{2}c\right)^{2}\right]=\frac{q}{1+q}(2c^{2}-1)(x-a)^{2}\left\|{}_{a}\mathrm{D}_{q}f\right\|。\tag{11}$$

对任意自然数k,有

$$f\left(q^{k}x+(1-q^{k})b\right)-f(x)=\sum_{m=0}^{k-1}\left[f\left(q^{m+1}x+(1-q^{m+1})b\right)-f\left(q^{m}x+(1-q^{m})b\right)\right]=(1-q)(b-x)\sum_{m=0}^{k-1}{}^{b}\mathrm{D}_{q}f\left(q^{m}x+(1-q^{m})b\right)q^{m}\leqslant(1-q)(b-x)\sum_{m=0}^{k-1}\left\|{}^{b}\mathrm{D}_{q}f\right\|q^{m}=(1-q^{k})(b-x)\left\|{}^{b}\mathrm{D}_{q}f\right\|,$$

$$f\left(q^{k}x+(1-q^{k})b\right)-f(b)=\sum_{m=k}^{\infty}\left[f\left(q^{m}x+(1-q^{m})b\right)-f\left(q^{m+1}x+(1-q^{m+1})b\right)\right]=(1-q)(b-x)\sum_{m=k}^{\infty}\left[-{}^{b}\mathrm{D}_{q}f\left(q^{m}x+(1-q^{m})b\right)\right]q^{m}\leqslant(1-q)(b-x)\sum_{m=k}^{\infty}\left\|{}^{b}\mathrm{D}_{q}f\right\|q^{m}=q^{k}(b-x)\left\|{}^{b}\mathrm{D}_{q}f\right\|,$$

$$f\left(q^{k}x+(1-q^{k})b\right)\leqslant\min\left\{f(b)+q^{k}(b-x)\left\|{}^{b}\mathrm{D}_{q}f\right\|,\right.$$

$$f(x)+(1-q^k)(b-x)\left\|{}^b\mathrm{D}_q f\right\|\Big\}=:\sigma,$$

$$\left[f(b)+q^k(b-x)\left\|{}^b\mathrm{D}_q f\right\|\right]-$$

$$\left[f(x)+(1-q^k)(b-x)\left\|{}^b\mathrm{D}_q f\right\|\right]=$$

$$2(q^k-d)(b-x)\left\|{}^b\mathrm{D}_q f\right\|,$$

因为

$$|f(x)-f(b)|=$$

$$\left|\sum_{m=0}^{\infty}\left[f\left(q^m x+(1-q^m)b\right)-f\left(q^{m+1}x+(1-q^{m+1})b\right)\right]\right|=$$

$$(1-q)(b-x)\left|\sum_{m=0}^{\infty}{}^b\mathrm{D}_q f\left(q^m x+(1-q^m)b\right)q^m\right|\leqslant$$

$$(1-q)(b-x)\sum_{m=0}^{\infty}\left|{}^b\mathrm{D}_q f\left(q^m x+(1-q^m)b\right)\right|q^m\leqslant$$

$$(1-q)(b-x)\sum_{m=0}^{\infty}\left\|{}^b\mathrm{D}_q f\right\|q^m=(b-x)\left\|{}^b\mathrm{D}_q f\right\|,$$

故有 $d\in[0,1]$,从而

$$\sigma=\begin{cases}f(x)+(1-q^k)(b-x)\left\|{}^b\mathrm{D}_q f\right\|, & 0\leqslant k\leqslant L;\\ f(b)+q^k(b-x)\left\|{}^b\mathrm{D}_q f\right\|, & L+1\leqslant k<\infty。\end{cases}$$

式中 $L=\left[\log_q d\right]$。

$$\int_x^b f(t)\,{}_a\mathrm{d}_q t=(1-q)(b-x)\left[\sum_{k=0}^{L}q^k f\left(q^k x+(1-q^k)b\right)+\right.$$

$$\left.\sum_{k=L+1}^{\infty}q^k f\left(q^k x+(1-q^k)b\right)\right]\leqslant$$

$$(1-q)(b-x)\left\{\sum_{k=0}^{L}q^k\left[f(x)+(1-q^k)(b-x)\left\|{}^b\mathrm{D}_q f\right\|\right]+\right.$$

$$\left.\sum_{k=L+1}^{\infty}q^k\left[f(b)+q^k(b-x)\left\|{}^b\mathrm{D}_q f\right\|\right]\right\}=$$

$$(b-x)\left[(1-q^{L+1})f(x)+\left(1-q^{L+1}-\frac{1-2q^{2(L+1)}}{1+q}\right)\cdot\right.$$

$$\left.(b-x)\left\|{}^b\mathrm{D}_q f\right\|+q^{L+1}f(b)\right],$$

$$(b-x)f(x)-\int_x^b f(t)\,{}^b\mathrm{d}_q t\geqslant$$

$$(b-x)^2\left\|{}^b\mathrm{D}_q f\right\|\left[\frac{1+q}{2}d^2-\frac{q}{1+q}-\right.$$

$$\left.\frac{2}{1+q}\left(q^{L+1}-\frac{1+q}{2}d\right)^2\right],$$

由 $\log_q d-1<L\leqslant\log_q d$,得 $d>q^{L+1}\geqslant qd$,从而有 $\left|q^{L+1}-\frac{1+q}{2}d\right|\leqslant\frac{1-q}{2}d$,于是有

$$(b-x)f(x)-\int_x^b f(t)\,{}^b\mathrm{d}_q t\geqslant$$

$$(b-x)^2\left\|{}^b\mathrm{D}_q f\right\|\left[\frac{1+q}{2}d^2-\frac{q}{1+q}-\frac{2}{1+q}\left(\frac{1-q}{2}d\right)^2\right]=$$

$$\frac{q}{1+q}(2d^2-1)(b-x)^2\left\|{}^b\mathrm{D}_q f\right\|。\qquad(12)$$

将式(11)与式(12)相加,则式(10)的左边不等式得证。对$(-f)$应用已证结果,则式(10)的右边不等式得证。

推论3　设f是定义在$[a,b]$上的函数,且f在点$x=a$处和点$x=b$处连续,$\left\|{}_a\mathrm{D}_q f\right\|$和$\left\|{}^b\mathrm{D}_q f\right\|$存在有限且非零,则有

$$|J|\leqslant\frac{q}{(1+q)(b-a)}\left[(x-a)^2\left\|{}_a\mathrm{D}_q f\right\|+(b-x)^2\left\|{}^b\mathrm{D}_q f\right\|\right]。$$

定理6　设f是$[a,b]$上的二阶q可微函数和二阶q^b可微函数,$\left\|{}_a\mathrm{D}_q^2 f\right\|\leqslant M$,$\left\|{}^b\mathrm{D}_q^2 f\right\|\leqslant M$,则有

$$|H|\leqslant\frac{q^3M}{1+q}\left\{\frac{(x-a)^3+(b-x)^3}{1+q+q^2}-\frac{\left(1+q+q^2\right)^2}{27\left(1+q+q^2-q^3\right)^2}\cdot\right.$$

$$\left.\left[(x-a)^3\left(1-\frac{|S_1|}{M}\right)^3+(b-x)^3\left(1-\frac{|S_2|}{M}\right)^3\right]\right\}。$$

$$(13)$$

证明　由式(2)~(3)得

$$H=\frac{q^3}{1+q}\left[(x-a)^3L_1+(b-x)^3L_2\right],$$

$$|H|\leqslant\frac{q^3}{1+q}\left[(x-a)^3|L_1|+(b-x)^3|L_2|\right],\qquad(14)$$

对任意常数$\varepsilon\in(0,1)$,有

$$|L_1|=\left|\int_0^1\left(t^2-\varepsilon^2\right){}_a\mathrm{D}_q^2 f\left(tx+(1-t)a\right)\mathrm{d}_q t+\right.$$

$$\left.\varepsilon^2\int_0^1{}_a\mathrm{D}_q^2 f\left(tx+(1-t)a\right)\mathrm{d}_q t\right|\leqslant$$

$$\int_0^1\left|t^2-\varepsilon^2\right|\left\|{}_a\mathrm{D}_q^2 f\left(tx+(1-t)a\right)\right|\mathrm{d}_q t+$$

$$\varepsilon^2\left|\int_0^1{}_a\mathrm{D}_q^2 f\left(tx+(1-t)a\right)\mathrm{d}_q t\right|\leqslant$$

$$M\int_0^1\left|t^2-\varepsilon^2\right|\mathrm{d}_q t+\varepsilon^2|S_1|,$$

$$\int_0^1\left|t^2-\varepsilon^2\right|\mathrm{d}_q t=(1-q)\sum_{k=0}^{\infty}q^k\left|q^{2k}-\varepsilon^2\right|=$$

$$(1-q)\left[\sum_{k=0}^{K}q^k\left(q^{2k}-\varepsilon^2\right)+\sum_{k=K+1}^{\infty}q^k\left(\varepsilon^2-q^{2k}\right)\right]=$$

$$\frac{1-2q^{3(K+1)}}{1+q+q^2}+\varepsilon^2\left(2q^{K+1}-1\right),$$

式中 $K=\left[\log_q\varepsilon\right]$。由 $\log_q\varepsilon-1<K\leqslant\log_q\varepsilon$ 得 $\varepsilon>q^{K+1}\geqslant q\varepsilon$,综上所述得

$$\left|L_1\right|\leqslant M\left[\frac{1-2q^{3(K+1)}}{1+q+q^2}+\varepsilon^2\left(2q^{K+1}-1\right)\right]+\varepsilon^2\left|S_1\right|\leqslant$$

$$M\left[\frac{1-2q^3\varepsilon^3}{1+q+q^2}+\varepsilon^2(2\varepsilon-1)\right]+\varepsilon^2\left|S_1\right|=:\varphi(\varepsilon),\tag{15}$$

$$\varphi'(\varepsilon)=6\left(1-\frac{q^3}{1+q+q^2}\right)\varepsilon\left(\varepsilon-\varepsilon_0\right),$$

式中 $\varepsilon_0=\dfrac{1+q+q^2}{3\left(1+q+q^2-q^3\right)}\left(1-\dfrac{\left|S_1\right|}{M}\right)$。因为 $\left|S_1\right|\leqslant M$,故有 $0\leqslant\varepsilon_0\leqslant\dfrac{1+q+q^2}{3\left(1+q+q^2-q^3\right)}<1$。当 $\varepsilon_0>0$ 时,在式(15)中取 $\varepsilon=\varepsilon_0$ 得

$$\left|L_1\right|\leqslant\varphi\left(\varepsilon_0\right)=\varphi(0)+\int_0^{\varepsilon_0}\varphi'(\varepsilon)\,\mathrm{d}\varepsilon=\frac{M}{1+q+q^2}+$$

$$\frac{6\left(1+q+q^2-q^3\right)}{1+q+q^2}\int_0^{\varepsilon_0}\varepsilon\left(\varepsilon-\varepsilon_0\right)\mathrm{d}\varepsilon=$$

$$\frac{M}{1+q+q^2}-\frac{\left(1+q+q^2\right)^2M}{27\left(1+q+q^2-q^3\right)^2}\left(1-\frac{\left|S_1\right|}{M}\right)^3,\tag{16}$$

当 $\varepsilon_0=0$ 时,$\left|S_1\right|=M$,在式(15)中令 $\varepsilon\to0$ 得 $\left|L_1\right|\leqslant\dfrac{M}{1+q+q^2}$,即式(16)也成立。类似可证

$$\left|L_2\right|\leqslant\frac{M}{1+q+q^2}-\frac{\left(1+q+q^2\right)^2M}{27\left(1+q+q^2-q^3\right)^2}\left(1-\frac{\left|S_2\right|}{M}\right)^3,\tag{17}$$

综合式(14),式(16)~(17),则式(13)得证。

推论4 设f是$[a,b]$上的二阶可微函数,$\left\|f''\right\|_\infty=\sup\limits_{t\in(a,b)}\left|f''(t)\right|<\infty$,则对任意$x\in[a,b]$有

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant$$

$$\frac{\left\|f''\right\|_\infty}{2}\left\{\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2-\right.$$

$$\frac{1}{12(b-a)}\left[(x-a)^3\left(1-\frac{f'(x)-f'(a)}{(x-a)M}\right)^3+\right.$$

$$\left.\left.(b-x)^3\left(1-\frac{f'(b)-f'(x)}{(b-x)M}\right)^3\right]\right\}。\tag{18}$$

证明 在定理6中令 $q\to1$,注意到 ${}_a\mathrm{D}_q^2f(x)\to f''(x)$,${}^b\mathrm{D}_q^2f(x)\to f''(x)$,

$$\frac{q^2+q-1}{1-q^2}(b-a)f(x)-$$

$$\frac{q}{1-q^2}\left[(x-a)f\left(qx+(1-q)a\right)+(b-x)f\left(qx+(1-q)b\right)\right]=$$

$$-(b-a)f(x)-\frac{q}{1+q}\left[(b-x)^2\frac{f\left(qx+(1-q)b\right)-f(x)}{(1-q)(b-x)}-\right.$$

$$\left.(x-a)^2\frac{f\left(qx+(1-q)a\right)-f(x)}{(1-q)(x-a)}\right]\to-(b-a)f(x)-$$

$$(b-a)\left(\frac{a+b}{2}-x\right)f'(x),$$

取 $M=\left\|f''\right\|_\infty$,由式(13)即可证得式(18)。

注3 推论4是文献[3]建立不等式的加强。

定理7 设f是$[a,b]$上的二阶q可微函数和二阶q^b可微函数,$\left\|{}_a\mathrm{D}_q^2f\right\|\leqslant M$,$\left\|{}^b\mathrm{D}_q^2f\right\|\leqslant M$,则有

$$\left|{}_a^bL_q(x)\right|\leqslant\frac{(x-a)^2(b-x)^2(b-a)M}{1+q+q^2}\cdot$$

$$\left\{1-\frac{\left(1+q+q^2\right)^3}{27(1+q+q^3)^2(b-a)}\cdot\right.$$

$$\left.\left[(x-a)\left(1-\frac{\left|S_1\right|}{M}\right)^3+(b-x)\left(1-\frac{\left|S_2\right|}{M}\right)^3\right]\right\}。\tag{18}$$

证明 利用式(2)~(3)得

$${}_a^bL_q(x)=(x-a)^2(b-x)^2\left[(a-x)L_1+(x-b)L_2\right],$$

$$\left|{}_a^bL_q(x)\right|\leqslant(x-a)^2(b-x)^2\left[(x-a)\left|L_1\right|+(b-x)\left|L_2\right|\right],\tag{19}$$

综合式(19),式(16)~(17),则式(18)得证。

注4 定理7给出式(4)的加强。

2 结束语

本文建立了若干量子Ostrowski型不等式,在已有文献给出的不等式右边减去了一个非负项,显然加强了已有的结果。当$q\to1$时由本文结果得到经典的Ostrowski不等式和Ostrowski型不等式。在定理3~5的证明中,先利用量子平均值不等式将定积分表示为若干部分之和,然后对每个部分进行放大。定理6的证明仿照了文献[27]引入参数求最值的方法。

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