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Pell数列偶数项与奇数项的倒数和*

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摘要:文章根据Pell数列的通项公式,运用初等方法研究了Pell数列中奇数项与偶数项的倒数和,得到了Pell数列偶数项和奇数项的倒数和的2个恒等式。

关键词:Pell数列;奇数项;偶数项;倒数和
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Reciprocal sum of even and odd terms in Pell number*

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Abstract: The reciprocal sums of the odd and even terms in Pell number were searched by using the elementary method, and obtained two identities of the reciprocal sum of the even and odd terms in Pell sequence.

Keywords: Pell number; odd number; even number; reciprocal sum
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0　引　言

著名的Fibonacci数列

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1, n \geqslant 2$$

和Lucas数列

$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1, n \geqslant 2,$$

满足二次线性的递推关系,这2个数列在初等数论的理论研究中有重要的作用,不少学者对这2个数列的一些性质进行了深入细致的研究,并且取得了不少的研究成果。

在文献[1-5]中,学者研究了Fibonacci数列倒数的一些和的恒等式:Ohtsuka和Nakamuras[1]研究了

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, n\text{是偶数}, \\ F_{n-2} - 1, n\text{是奇数}, \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, n\text{是偶数}, \\ F_n F_{n-1}, n\text{是奇数}; \end{cases}$$

吴振刚和王婷婷[4]证明了Fibonacci数列倒数的有限和

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2}。$$

在文献[6-10]中,学者讨论了Fibonacci数列和Lucas数列的一些性质,如杨海等[6]和晁晶晶[7]探讨了Fibonacci数与杨辉三角形的关系等。

Pell数列与Fibonacci数列类似,是由二阶线性递归序列 $P_{n+2} = 2P_{n+1} + P_n$ 定义的,其中 $P_0 = 0, P_1 = 1$。由特征方程 $x^2 - x - 1 = 0$ 可以得到Pell数列的通项公式为[1]

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$$P_n=\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]。$$

一些学者研究了Pell数列的一些性质[10-15],讨论了Pell数列倒数的一些和的恒等式[16-20]。如:Zhang和Wang[17]研究了Pell倒数的无限和

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{P_k}\right)^{-1}\right\rfloor=\begin{cases}P_{n-1}+P_{n-2},n为偶数,\\P_{n-1}+P_{n-2}-1,n为奇数;\end{cases}$$

张文鹏和王婷婷[18]研究了Pell平方倒数的无限和

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{P_k^2}\right)^{-1}\right\rfloor=\begin{cases}2P_{n-1}P_n-1,n为偶数,\\2P_{n-1}P_n,n为奇数;\end{cases}$$

Xu和Wang[19]研究了Pell立方倒数的无限和

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{P_k^3}\right)^{-1}\right\rfloor=$$

$$\begin{cases}P_n^2P_{n-1}+3P_nP_{n-1}^2+\left\lfloor-\frac{61}{82}P_n-\frac{91}{82}P_{n-1}\right\rfloor,n为偶数,\\P_n^2P_{n-1}+3P_nP_{n-1}^2+\left\lfloor\frac{61}{82}P_n+\frac{91}{82}P_{n-1}\right\rfloor,n为奇数。\end{cases}$$

本文运用初等方法给出Pell数列偶数项和奇数项的倒数和的2个恒等式:

定理1 当$n\geqslant 1,m\geqslant 3$时,

$$\left\lfloor\left(\sum_{k=n}^{mn}\frac{1}{P_{2k}}\right)^{-1}\right\rfloor=2P_{2n-1}-1。$$

定理2 当$n\geqslant 1,m\geqslant 2$时,

$$\left\lfloor\left(\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}\right)^{-1}\right\rfloor=2P_{2n-2}。$$

1 有关引理

引理1 $P_mP_n-P_sP_t=(-1)^{m+1}P_{n-s}P_{n-t},(n>\max\{s,t\},m+n=s+t)$。

证明 由于$P_n=\frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n-(1-\sqrt{2})^n\right]$,令$1+\sqrt{2}=\alpha,1-\sqrt{2}=\beta$,则$\alpha\beta=-1$,于是

$$P_mP_n-P_sP_t=\left[\frac{1}{2\sqrt{2}}(\alpha^m-\beta^m)\right]\left[\frac{1}{2\sqrt{2}}(\alpha^n-\beta^n)\right]-\left[\frac{1}{2\sqrt{2}}(\alpha^s-\beta^s)\right]\left[\frac{1}{2\sqrt{2}}(\alpha^t-\beta^t)\right]=$$

$$\frac{1}{8}\left[(\alpha^{m+n}-\alpha^n\beta^m-\alpha^m\beta^n+\beta^{m+n})-(\alpha^{s+t}-\alpha^s\beta^t-\alpha^t\beta^s+\beta^{s+t})\right]=$$

$$\frac{1}{8}(\alpha\beta)^m(\alpha^{s-m}\beta^{t-m}+\alpha^{t-m}\beta^{s-m}-\beta^{n-m}-\alpha^{n-m})=$$

$$\frac{1}{8}(-1)^{m+1}(\beta^{n-m}+\alpha^{n-m}-\alpha^{s-m}\beta^{t-m}-\alpha^{t-m}\beta^{s-m})=$$

$$\frac{1}{8}(\alpha\beta)^{m+1}(\beta^{n-(s+t-n)}+\alpha^{n-(s+t-n)}-\alpha^{n-t}\beta^{n-s}-\alpha^{n-s}\beta^{n-t})=$$

$$(-1)^{m+1}P_{n-s}P_{n-t}。$$

引理2 $P_mP_n+P_{m+1}P_{n+1}=P_{m+n+1}$。

证明 令$1+\sqrt{2}=\alpha,1-\sqrt{2}=\beta$,则$\alpha\beta=-1$,

$$\frac{1}{\alpha}=\frac{1}{1+\sqrt{2}}=-(1-\sqrt{2})=\sqrt{2}-1,$$

$$\frac{1}{\beta}=\frac{1}{1-\sqrt{2}}=-(1+\sqrt{2})=-1-\sqrt{2},$$

$$P_mP_n+P_{m+1}P_{n+1}=$$

$$\left[\frac{1}{2\sqrt{2}}(\alpha^m-\beta^m)\right]\left[\frac{1}{2\sqrt{2}}(\alpha^n-\beta^n)\right]+\left[\frac{1}{2\sqrt{2}}(\alpha^{m+1}-\beta^{m+1})\right]\left[\frac{1}{2\sqrt{2}}(\alpha^{n+1}-\beta^{n+1})\right]=$$

$$\frac{1}{8}(\alpha^{m+n}-\alpha^m\beta^n-\alpha^n\beta^m+\beta^{m+n}+\alpha^{m+n+2}-\alpha^{m+1}\beta^{n+1}-\alpha^{n+1}\beta^{m+1}+\beta^{m+n+2})=$$

$$\frac{1}{2\sqrt{2}}(\alpha^{m+n+1}-\beta^{m+n+1})=P_{m+n+1}。$$

2 主要结果

定理1 当$n\geqslant 1,m\geqslant 3$时,有

$$\left\lfloor\left(\sum_{k=n}^{mn}\frac{1}{P_{2k}}\right)^{-1}\right\rfloor=2P_{2n-1}-1。$$

证明 由引理1可得

$$\frac{1}{2P_{2k-1}-1}-\frac{1}{P_{2k}}-\frac{1}{2P_{2k+1}-1}=\frac{P_{2k}(2P_{2k+1}-1)-(2P_{2k-1}-1)(2P_{2k+1}-1)-P_{2k}(2P_{2k-1}-1)}{(2P_{2k-1}-1)P_{2k}(2P_{2k+1}-1)}=$$

$$\frac{2P_{2k+1}P_{2k-2}-2P_{2k}P_{2k-1}+2P_{2k-1}+2P_{2k+1}-1}{(2P_{2k-1}-1)P_{2k}(2P_{2k-2}-1)}=$$

$$\frac{2P_{2k-1}+2P_{2k+1}-5}{P_{2k}(2P_{2k-1}-1)(2P_{2k+1}-1)},\qquad(1)$$

因此有

$$\sum_{k=n}^{mn}\frac{1}{P_{2k}}=\sum_{k=n}^{mn}\frac{1}{2P_{2k-1}-1}-\sum_{k=n}^{mn}\frac{1}{2P_{2k+1}-1}-\sum_{k=n}^{mn}\frac{2P_{2k-1}+2P_{2k+1}-5}{P_{2k}(2P_{2k-1}-1)(2P_{2k+1}-1)}=$$

$$\frac{1}{2P_{2n-1}-1}-\frac{1}{2P_{2mn+1}-1}-\sum_{k=n}^{mn}\frac{2P_{2k-1}+2P_{2k-1}-5}{P_{2k}\left(2P_{2k-1}-1\right)\left(2P_{2k+1}-1\right)}。$$

所以

$$\sum_{k=n}^{mn}\frac{1}{P_{2k}}<\frac{1}{2P_{2n-1}-1}。\tag{2}$$

由引理1可知

$$\frac{1}{2P_{2k-1}}-\frac{1}{P_{2k}}-\frac{1}{2P_{2k+1}}=\frac{2P_{2k}P_{2k+1}-4P_{2k-1}P_{2k+1}-2P_{2k}P_{2k-1}}{4P_{2k-1}P_{2k}P_{2k+1}}=\frac{-4}{4P_{2k-1}P_{2k}P_{2k+1}}=-\frac{1}{P_{2k-1}P_{2k}P_{2k+1}}。\tag{3}$$

则

$$\sum_{k=n}^{mn}\frac{1}{P_{2k}}=\sum_{k=n}^{mn}\frac{1}{2P_{2k-1}}-\sum_{k=n}^{mn}\frac{1}{2P_{2k+1}}+\sum_{k=n}^{mn}\frac{1}{P_{2k-1}P_{2k}P_{2k+1}}=\frac{1}{2P_{2n-1}}-\frac{1}{2P_{2mn+1}}+\sum_{k=n}^{mn}\frac{1}{P_{2k}P_{2k-1}P_{2k+1}}>\frac{1}{2P_{2n-1}}-\frac{1}{2P_{2mn+1}}+\frac{1}{P_{2n-1}P_{2n}P_{2n+1}}。$$

当$n\geqslant 1,m\geqslant 3$时,$P_{2n-1}P_{2n}P_{2n+1}<2P_{2mn+1}$。由引理2可得

$$P_{m-1}P_n+P_mP_{n+1}=P_{m+n}。$$

因此

$$P_{m+n}\geqslant P_mP_{n+1}\geqslant P_mP_n,$$
$$P_{2n-1}P_{2n}P_{2n+1}<P_{6n+1}<P_{2mn+1}。$$

故有

$$\sum_{k=n}^{mn}\frac{1}{P_{2k}}>\frac{1}{P_{2n-1}}。\tag{4}$$

由式(2)和(4)可得定理1。

推论1　$n\geqslant 1,m\geqslant 3$时,

$$\left[\sum_{k=n}^{\infty}\left(\frac{1}{P_{2k}}\right)^{-1}\right]=2P_{2n-2}-1。$$

证明　由式(1)可知

$$\frac{1}{P_{2k}}=\frac{1}{2P_{2k-1}-1}-\frac{1}{2P_{2k+1}-1}-\frac{2P_{2k-1}-2P_{2k+1}-5}{\left(2P_{2k-1}-1\right)P_{2k}\left(2P_{2k+1}-1\right)}<\frac{1}{2P_{2k-1}-1}-\frac{1}{2P_{2k+1}-1},$$

因而

$$\sum_{k=n}^{\infty}\frac{1}{P_{2k}}<\sum_{k=n}^{\infty}\left(\frac{1}{2P_{2k-1}-1}-\frac{1}{2P_{2k+1}-1}\right)<\frac{1}{2P_{2n-1}-1},\tag{5}$$

由(3)可知

$$\frac{1}{P_{2k}}=\frac{1}{2P_{2k-1}}-\frac{1}{2P_{2k+1}}+\frac{1}{P_{2k-1}P_{2k}P_{2k+1}},$$
$$\sum_{k=n}^{\infty}\frac{1}{P_{2k}}=\sum_{k=n}^{\infty}\frac{1}{2P_{2k-1}}-\sum_{k=n}^{\infty}\frac{1}{2P_{2k+1}}+\sum_{k=n}^{\infty}\frac{1}{P_{2k-1}P_{2k}P_{2k+1}}=\frac{1}{2P_{2n-1}}-\frac{1}{2P_{2n+1}}-\sum_{k=n}^{\infty}\frac{1}{P_{2k-1}P_{2k}P_{2k+1}}>\frac{1}{2P_{2n-1}}-\frac{1}{2P_{2n+1}},$$

所以

$$\sum_{k=n}^{\infty}\frac{1}{P_{2k}}>\frac{1}{2P_{2n-1}}。\tag{6}$$

由式(5)和(6)可得推论1。

定理2　当$n\geqslant 1,m\geqslant 2$,有

$$\left[\left(\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}\right)^{-1}\right]=2P_{2n-2}。$$

证明　引理1可得

$$\frac{1}{2P_{2k-2}}-\frac{1}{P_{2k-1}}-\frac{1}{2P_{2k}}=\frac{2P_{2k-1}P_{2k}-4P_{2k}P_{2k-2}-2P_{2k-1}P_{2k-2}}{4P_{2k-2}P_{2k-1}P_{2k}}=\frac{2P_{2k}P_{2k-3}-2P_{2k-2}P_{2k-1}}{4P_{2k-2}P_{2k-2}P_{2k}}=\frac{1}{P_{2k-2}P_{2k-1}P_{2k}}>0,$$

则可得

$$\sum_{k=n}^{mn}\frac{1}{2P_{2k-1}}<\frac{1}{2P_{2n-2}}-\frac{1}{2P_{2mn}}<\frac{1}{2P_{2n-2}}。\tag{7}$$

同理有

$$\frac{1}{2P_{2k-2}+1}-\frac{1}{P_{2k}}-\frac{1}{2P_{2k+1}}=\frac{P_{2k-1}\left(2P_{2k}+1\right)-\left(2P_{2k-2}+1\right)\left(2P_{2k}+1\right)-P_{2k-1}\left(2P_{2k-2}+1\right)}{\left(2P_{2k-2}+1\right)P_{2k-1}\left(2P_{2k}+1\right)}=\frac{2P_{2k}\left(P_{2k-1}-2P_{2k-2}\right)-2P_{2k-1}P_{2k-2}-2P_{2k-2}-2P_{2k}-1}{\left(2P_{2k-2}+1\right)P_{2k-1}\left(2P_{2k}+1\right)}=-\frac{2P_{2k-2}+2P_{2k}-3}{\left(2P_{2k-2}+1\right)P_{2k-1}\left(2P_{2k}+1\right)},$$

由上式可得

$$\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}=\sum_{k=n}^{mn}\frac{1}{2P_{2k-2}+1}-\sum_{k=n}^{mn}\frac{1}{2P_{2k}+1}+$$

$$\sum_{k=2}^{mn}\frac{2P_{2k-2}+2P_{2k-2}+2P_{2k}-3}{\left(2P_{2k-2}+1\right)P_{2k-1}\left(2P_{2k}+1\right)}=\frac{1}{2P_{2n-2}+1}-\frac{1}{2P_{2mn}+1}+\sum_{k=n}^{mn}\frac{2P_{2k-2}+2P_{2k}-3}{\left(2P_{2k-2}-1\right)\left(2P_{2k}+1\right)}>\frac{1}{2P_{2n-2}+1}-\frac{1}{2P_{2mn}+1}+\frac{2P_{2n-2}+2P_{2n}-3}{\left(2P_{2n-2}+1\right)P_{2n-1}\left(2P_{2n}+1\right)}。$$

由引理2可知

$$2P_{4n+1}\geqslant 2P_{2n+1}P_{2n}>\left(2P_{2n-2}+1\right)P_{2n-1}。$$

所以

$$\frac{2P_{2n}+2P_{2n-2}-3}{\left(2P_{2n-2}+1\right)P_{2n-1}\left(2P_{2n}+1\right)}>\frac{1}{\left(2P_{2n-2}+1\right)P_{2n-1}}>\frac{1}{2P_{4n+1}}\geqslant\frac{1}{2P_{mn}+1},$$

因此

$$\sum_{k=n}^{mm}\frac{1}{P_{2k-1}}>\frac{1}{2P_{2n-2}+1},\tag{8}$$

由式(7)和(8)可得

$$\frac{1}{2P_{2n-2}+1}<\sum_{k=n}^{mn}\frac{1}{P_{2k-1}}<\frac{1}{2P_{2n-2}}。$$

推论2 $n\geqslant 1, m\geqslant 2$

$$\left[\left(\sum_{k=n}^{\infty}\frac{1}{P_{2k}}\right)^{-1}\right]=2P_{2n-2},$$

$$\frac{1}{P_{2k-1}}=\frac{1}{2P_{2k-2}}-\frac{1}{2P_{2k}}-\frac{1}{P_{2k-2}P_{2k}P_{2k-1}}<\frac{1}{2P_{2k-2}}-\frac{1}{2P_{2k-1}}。$$

因此

$$\sum_{i=n}^{\infty}\frac{1}{P_{2k-1}}<\sum_{k=n}^{\infty}\left(\frac{1}{2P_{2k-2}}-\frac{1}{2P_{2k-1}}\right),$$

$$\sum_{k=n}^{mn}\frac{1}{P_{2k}}<\frac{1}{2P_{2n-1}}-\frac{1}{2P_{2n-1}}<\frac{1}{2P_{2n-2}},\tag{9}$$

$$\frac{1}{P_{2n-2}}=\frac{1}{2P_{2k-2}+1}-\frac{1}{2P_{2k}+1}+\frac{2P_{2k-2}-2P_{2k}+3}{\left(2P_{2n-2}+1\right)P_{2n-1}\left(2P_{2n}+1\right)},$$

$$\sum_{k=n}^{\infty}\frac{1}{P_{2k-1}}=\sum_{k=n}^{\infty}\frac{1}{2P_{2k-2}+1}-\sum_{k=n}^{\infty}\frac{1}{2P_{2k}+1}+\sum_{k=n}^{\infty}\frac{2P_{2k-2}+2P_{2k}-3}{\left(2P_{2k-2}+1\right)P_{2k-1}\left(2P_{2k}+1\right)},$$

$$\sum_{k=n}^{\infty}\frac{1}{P_{2k-1}}=\frac{1}{2P_{2n-2}+1}+\frac{2P_{2n-2}+2P_{2n}-3}{\left(2P_{2n-2}+1\right)P_{2n-1}\left(2P_{2n}+1\right)}>\frac{1}{2P_{2n-2}+1}。\tag{10}$$

由式(9)和(10)可得推论2。

3 结束语

文章根据Pell数列的通项公式,利用初等方法对Pell数列和的相关性质进行了初步的探究,给出了Pell数列偶数项和奇数项的倒数和的2个恒等式。对于Pell数列其他和的性质,比如Pell数列每项平方的倒数和,连续2项乘积的倒数和等有待进一步进行研究。

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