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有关模 n 简化剩余系一性质的另一证明

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摘 要

设 n 是正整数,$\varphi(n)$ 是 Euler 函数. 设 $M=\{a_1,a_2,\cdots,a_{\varphi(n)}\}$ 是模 n 的最小正简化剩余系,则 $\sum_{i=1}^{\varphi(n)}a_i=\left(\frac{n}{2}\right)\varphi(n)$. 针对这一性质,本文将给出其另一证明.

关键词: Euler 函数,简化剩余系,证明.
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1 引 言

设 n 是正整数,$\varphi(n)$ 是 Euler 函数,它在 n 上的值等于序列 $0,1,2,\cdots,n-1$ 中与 n 互素的整数的个数. Euler 函数 $\varphi(n)$ 是数论中的一个重要函数,它与 Smarandache 函数联系极为密切,有关这两个函数的研究可参看一些文献,例如文献[1-5]. 对于涉及 Euler 函数的性质以及与其有关方程的研究甚多. 例如文献[6-12]. 在由单墫教授主编的《初等数论》一书中给出了如下性质:

性质 所有小于 n 且与 n 互素的正整数之和为 $\left(\frac{n}{2}\right)\varphi(n)$ [13].

由于所有小于 n 且与 n 互素的正整数所构成的集合恰为模 n 的最小正简化剩余系,因而对于该猜想可以表述为:

性质 1 设 $M=\{a_1,a_2,\cdots,a_{\varphi(n)}\}$ 是模 n 最小正简化剩余系,则 $\sum_{i=1}^{\varphi(n)}a_i=\left(\frac{n}{2}\right)\varphi(n)$.

对于该性质,有关文献也进行了研究证明,如文献[14-15]. 本文将利用初等方法,给出了该性质的另一证明.

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2 符号说明

令 $n=P_1^{\alpha_2}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 为正整数 n 的标准分解式,其中 $P_i(i=1,2,\cdots,k)$ 是满足 $P_1<P_2<\cdots<P_k$ 的素数,$\alpha_i(i=1,2,\cdots,k)$ 是正整数. 现利用 n 的素因数 P_i 对 n 的因数进行分类,将分成 2^k-1 个集合,具体情况如下:

用 $\{P_i\}$ 表示为区间 $[1,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}]$ 内为素因数 P_i 倍数的整数所构成集合,这样的集合有 $C_k^1=k$ 个;

用 $\{P_i\cap P_j\}$ 表示为区间 $[1,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}]$ 内同时为素因数 P_i 与 P_j 倍数的整数所构成集合,其中 $1\leqslant i,j\leqslant k,i\neq j$,这样的集合有 $C_k^2=\frac{k(k-1)}{2}$ 个;

用 $\{P_i\cap P_j\cap P_r\}$ 表示为区间 $[1,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}]$ 内同时为素因数 P_i,P_j 与 P_r 倍数的整数所构成集合,其中 $1\leqslant i,j,r\leqslant k,i\neq j\neq r$,这样的集合有 $C_k^3=\frac{k(k-1)(k-2)}{3}$ 个;

依次类推,第 2^k-1 个集合用 $\{P_1\cap P_2\cap\cdots\cap P_k\}$ 表示为区间 $[1,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}]$ 内同时为素因数 $P_1,P_2,\cdots,P_k$ 倍数的整数所构成集合,这样的集合有 1 个;

用 S_{P_i} 表示为集合 $\{P_i\}$ 中所有整数的和;

用 $S_{P_i\cap P_j}$ 表示为集合 $\{P_i\cap P_j\}$ 中所有整数的和;

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用 $S_{P_i\cap P_j\cap P_r}$ 表示为集合$\{P_i \cap P_j \cap P_r\}$ 中所有整数的和;

依次类推,用 $S_{P_1\cap P_2\cap\cdots\cap P_k}$ 表示为集合$\{P_1 \cap P_2 \cap \cdots \cap P_k\}$ 中所有整数的和;

令 S 表示区间 $[\![,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}]$ 内所有整数的和,$S_1 = \sum_{i=1}^{\varphi(n)} a_i$,其中 $M = \{a_1,a_2,\cdots,a_{\varphi(n)}\}$ 是模 n 的最小正简化剩余系.

3　性质 1 的证明

对于任意大于 1 的正整数 $n = P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$,则有

$$\left(\frac{n}{2}\right)\varphi(n) = \frac{P_1^{2\alpha_1}P_2^{2\alpha_2}\cdots P_k^{2\alpha_k}}{2}\left(1-\frac{1}{P_1}\right)\cdot\left(1-\frac{1}{P_2}\right)\cdots\left(1-\frac{1}{P_k}\right),$$

$$S = \frac{\left(1+P_1^{\alpha_1}P_2^{\alpha_2}\cdots p_k^{\alpha_k}\right)P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2} = \frac{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}+p_1^{2\alpha_1}p_2^{2\alpha_2}\cdots p_k^{2\alpha_k}}{2}.$$

根据第 2 部分有关符号的说明可知,集合 $\{P_1\}$ 是由正整数 $P_1,2P_1,\cdots,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 构成的集合,集合 $\{P_1\}$ 中的整数个数为 $\left[\frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{P_1}\right] = P_1^{\alpha_1-1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$,其中符号 Ⅰ 为取整函数,则

$$S_{P_1} = \frac{P_1+P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2}P_1^{\alpha_1-1}P_2^{\alpha_2}\cdots P_k^{\alpha_k} = \frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}+P_1^{2\alpha_1-1}P_2^{2\alpha_2}\cdots P_k^{2\alpha_k}}{2};$$

集合 $\{P_2\}$ 是由正整数 $P_2,2P_2,\cdots,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 构成的集合,集合 $\{P_2\}$ 中的整数个数为 $\left[\frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{P_2}\right] = P_1^{\alpha_1}P_2^{\alpha_2-1}\cdots P_k^{\alpha_k}$,则

$$S_{P_2} = \frac{P_2+P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2}P_1^{\alpha_1}P_2^{\alpha_2-1}\cdots P_k^{\alpha_k} = \frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}+P_1^{2\alpha_1}P_2^{2\alpha_2-1}\cdots P_k^{2\alpha_k}}{2};$$

依次可得,集合 $\{P_k\}$ 是由正整数 $P_k,2P_k,\cdots,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 构成的集合,集合 $\{P_k\}$ 中的整数个数为 $\left[\frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{P_k}\right] = P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k-1}$,则

$$S_{P_k} = \frac{P_k+P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2}P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k-1} = \frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}+P_1^{2\alpha_1}P_2^{2\alpha_2}\cdots P_k^{2\alpha_k-1}}{2};$$

集合 $\{P_1P_2\}$ 是由正整数 $P_1P_2,2P_1P_2,\cdots,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 构成的集合,集合 $\{P_1P_2\}$ 中的整数个数为 $\left[\frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{P_1P_2}\right] = P_1^{\alpha_1-1}P_2^{\alpha_2-1}\cdots P_k^{\alpha_k}$,则

$$S_{P_1P_2} = \frac{P_1P_2+P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2}P_1^{\alpha_1-1}P_2^{\alpha_2-1}\cdots P_k^{\alpha_k} = \frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}+P_1^{2\alpha_1-1}P_2^{2\alpha_2-1}\cdots P_k^{2\alpha_k}}{2};$$

相类似的,集合 $\{P_{k-1}P_k\}$ 是由正整数 $P_{k-1}P_k,2P_{k-1}P_k,\cdots,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 构成的集合,集合 $\{P_{k-1}P_k\}$ 中的整数个数为 $\left[\frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{P_{k-1}P_k}\right] = P_1^{\alpha_1}\cdots P_{k-1}^{\alpha_{k-1}-1}P_k^{\alpha_k-1}$,则

$$S_{P_{k-1}P_k} = \frac{P_{k-1}P_k+P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2}P_1^{\alpha_1}\cdots P_{k-1}^{\alpha_{k-1}-1}P_k^{\alpha_k-1} = \frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}+P_1^{2\alpha_1}\cdots P_{k-1}^{2\alpha_{k-1}-1}P_k^{2\alpha_k-1}}{2};$$

集合 $\{P_1P_2P_3\}$ 是由正整数 $P_1P_2P_3,2P_1P_2P_3,\cdots,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 构成的集合,集合 $\{P_1P_2P_3\}$ 中的整数个数为 $\left[\frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{P_1P_2P_3}\right] = P_1^{\alpha_1-1}P_2^{\alpha_2-1}P_3^{\alpha_3-1}\cdots P_k^{\alpha_k}$,则

$$S_{P_1P_2P_3} = \frac{P_1P_2P_3+P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2}P_1^{\alpha_1-1}P_2^{\alpha_2-1}P_3^{\alpha_3-1}\cdots P_k^{\alpha_k} = \frac{P_1^{\alpha_1}P_2^{\alpha_2}P_3^{\alpha_3}\cdots P_k^{\alpha_k}+P_1^{2\alpha_1-1}P_2^{2\alpha_2-1}P_3^{2\alpha_3-1}\cdots P_k^{2\alpha_k}}{2};$$

类似的依次推进,集合 $\{P_1P_2\cdots P_k\}$ 是由整数 $P_1P_2\cdots P_k,2P_1P_2\cdots P_k,\cdots,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 构成的集合,集合 $\{P_1P_2\cdots P_k\}$ 中整数的个数为 $\left[\frac{P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{P_1P_2\cdots P_k}\right] = P_1^{\alpha_1-1}P_2^{\alpha_2-1}\cdots P_k^{\alpha_k-1}$,则

$$S_{P_1P_2\cdots P_k} = \frac{P_1P_2\cdots P_k+P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}}{2}P_1^{\alpha_1-1}P_2^{\alpha_2-1}\cdots P_k^{\alpha_k-1} = \frac{P_1^{\alpha_1}P_2^{\alpha_2}P_3^{\alpha_3}\cdots P_k^{\alpha_k}+P_1^{2\alpha_1-1}P_2^{2\alpha_2-1}\cdots P_k^{2\alpha_k-1}}{2}.$$

性质 1 的实质就是将区间 $[\![,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}]$ 内所有整数的和减去区间 $[\![,P_1^{\alpha_1}P_2^{\alpha_2}\cdots P_k^{\alpha_k}]$ 内所有与 $n = P_1^{\alpha_2}P_2^{\alpha_2}\cdots P_k^{\alpha_k}$ 不互素的整数的和. 在以上的 $\{P_1\}$, $\{P_2\}$,$\cdots$, $\{P_k\}$, $\{P_1P_2\}$,$\cdots$, $\{P_{k-1}P_k\}$, $\{P_1P_2P_3\}$,$\cdots$,$\{P_{k-2}P_{k-1}P_k\}$,$\cdots$,$\{P_1P_2\cdots P_k\}$ 这 2^k-1 个集合中,它们两两之间的交集不是空集,除非这里的 n 为素数或者 $n = P^\alpha$(P 为素数). 当 n 为素数或

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者 $n = P^{\alpha}$(P 为素数)时,可以很快的验证出性质是　　成立的. 因而有

$$S_1 = S + (-1)^1(\underbrace{S_{P_1} + S_{P_2} + \cdots + S_{P_k}}_{\text{以单个素因数考虑}}) + (-1)^2(\underbrace{S_{P_1P_2} + \cdots + S_{P_1P_k} + \cdots + S_{P_{k-1}P_k}}_{\text{以2个素因数考虑}}) +$$
$$(-1)^3(\underbrace{S_{P_1P_2P_3} + \cdots + S_{P_1P_2P_k} + \cdots + S_{P_{k-2}S_{P_{k-1}P_k}}}_{\text{以3个素因数考虑}}) + \cdots + (-1)^{k-1}(\underbrace{S_{P_1P_2\cdots P_{k-1}} + \cdots + S_{P_2\cdots P_{k-1}P_k}}_{\text{以}k-1\text{个素因数考虑}}) +$$
$$(-1)^k \underbrace{S_{P_1P_2P_3\cdots P_k}}_{\text{以}k\text{个素因数考虑}}$$
$$= \frac{{P_1}^{2\alpha_1}{P_2}^{2\alpha_2}\cdots{P_k}^{2\alpha_k}}{2}\Big(1 + \underbrace{(-1)^1\frac{1}{P_1} + (-1)^1\frac{1}{P_2} + \cdots + (-1)^1\frac{1}{P_k}}_{\text{以单个素因数考虑}} +$$
$$\underbrace{(-1)^2\frac{1}{P_1P_2} + (-1)^2\frac{1}{P_1P_3} + \cdots + (-1)^2\frac{1}{P_{k-1}P_k}}_{\text{以2个素因数考虑}} +$$
$$\underbrace{(-1)^3\frac{1}{P_1P_2P_3} + (-1)^3\frac{1}{P_1P_2P_4} + \cdots + (-1)^3\frac{1}{P_1P_2P_k} + \cdots + (-1)^3\frac{1}{P_{k-2}P_{k-1}P_k}}_{\text{以3个素因数考虑}} + \cdots +$$
$$\underbrace{(-1)^{k-1}\frac{1}{P_1P_2\cdots P_{k-1}} + (-1)^{k-1}\frac{1}{P_1P_3\cdots P_k} + \cdots + (-1)^{k-1}\frac{1}{P_2P_3\cdots P_k}}_{\text{以}k-1\text{个素因数考虑}} + \underbrace{(-1)^k\frac{1}{P_1P_2\cdots P_k}}_{\text{以}k\text{个素因数考虑}}\Big)$$
$$= \frac{{P_1}^{2\alpha_1}{P_2}^{2\alpha_2}\cdots{P_k}^{2\alpha_k}}{2}\Big(1 - \frac{1}{P_1}\Big)\Big(1 - \frac{1}{P_2}\Big)\cdots\Big(1 - \frac{1}{P_k}\Big) = \Big(\frac{n}{2}\Big)\varphi(n).$$ **证毕.**

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Another Proof of a Nature on Simplify Surplus System Module n

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Abstract

Let n be a positive integer, and let $\varphi(n)$ be Euler function. If $M = \{a_1, a_2, \cdots, a_{\varphi(n)}\}$ be a least positive simplify surplus system module n, then $\sum_{i=1}^{\varphi(n)} a_i = \left(\frac{n}{2}\right)\varphi(n)$. In this article, we will give another proof of it.

Key words: Euler function, simplify surplus system, proof.

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