DOI:10.19789/j.1004-9398.2019.06.009

首都师范大学学报(自然科学版)
第40卷 第6期 Journal of Capital Normal University No. 6
2019年12月 (Natural Science Edition) Dec.,2019

共轭解析函数的一个求法的注记*

黄 炜

(宝鸡职业技术学院,陕西 宝鸡 721013)

摘要: 给出了复变函数的共轭解析函数的充要条件、求法及其重要应用.

关键词: 解析函数; 共轭解析; 充要条件; 求法; 应用

中图分类号: O174.5

0 引 言

古老的复变函数起源于19世纪,从柯西起至今有150多年的历史,已发展出许多新分支、新领域和新方法,如今复变函数成为数学中不可或缺的重要组成部分. 解析函数是复变函数论起初所研究的主要对象(见文[1]). 文献[2]提出来共轭解析函数概念,与解析函数对称. 解析函数所能解决的所有问题共轭解析函数都可以用来解决,共轭解析函数比解析函数更直观方便. 不少文献[3-4]已对共轭解析函数进行了研究,本文在前面研究的基础上. 给出了共轭解析函数充要条件及求共轭解析函数一个新方法.

1 相关定义

定义1.1[1] 设函数 $w=f(z)$,$z\in D$; z_0,$z_0+\Delta z\in D$,如果极限 $\lim\limits_{\Delta z\to 0}\dfrac{\Delta w}{\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ 存在就说 $f(z)$ 在 z_0 可导,该极限值就称为 $f(z)$ 在 z_0 的导数,记作 $f'(z_0)$ 或 $\left.\dfrac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=z_0}$.

若 $f(z)$ 在 z_0 的 δ 邻域 (z_0,δ) 内可导,称 z_0 为解析点,否则称为奇点.

若 $f(z)$ 在 D 内处处解析,则称 $f(z)$ 在区域 D 内解析.

定义1.2[1] 比值 $\dfrac{\Delta w}{\Delta z}$ 的极限 $\lim\limits_{\Delta z\to 0}\dfrac{\Delta w}{\Delta z}=\lim\limits_{\Delta z\to 0}\dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ 存在,则就说 $f(z)$ 在 z_0 共轭可导,此极限值就称为 $f(z)$ 在 z_0 的共轭导数,记作 $f^{\rho}(z_0)$ 或 $\left.\dfrac{\mathrm{d}w}{\mathrm{d}\bar{z}}\right|_{z=z_0}$.

这时称函数 $w=f(z)$,$z\in D$ 于 z 点共轭可导或共轭可微[2].

2 结 论

定理1 设函数 $f(z)=u(x,y)+iv(x,y)$ 在区域 D 内共轭解析的充要条件是: 二元函数 $u(x,y)$ 和 $v(x,y)$ 在区域 D 内可微且满足 $\dfrac{\partial f}{\partial\bar{z}}=0$.

定理2 设函数 $f(z)=u(x,y)+iv(x,y)$ 在区域 D 内共轭解析,则:

$$f(\bar{z})=f(z)=u(z,0)+iv(z,0).$$

3 定理的证明

为了完成定理的证明需要下面的引理:

引理3.1 函数 $f(z)=u(x,y)+iv(x,y)$,则其形式导数为

$$\begin{cases}\dfrac{\partial f}{\partial z}=\dfrac{1}{2}\left(\dfrac{\partial f}{\partial x}-i\dfrac{\partial f}{\partial y}\right),\\[2ex]\dfrac{\partial f}{\partial\bar{z}}=\dfrac{1}{2}\left(\dfrac{\partial f}{\partial x}+i\dfrac{\partial f}{\partial y}\right).\end{cases}$$

证明:

设 $f=u+iv$,则 $\Delta f=f(z)-f(z_0)$,$\Delta f=\Delta u+i\Delta v$.

收稿日期: 2018-10-19

* 国家自然科学基金(10671155); 陕西省自然科学基金(2009JQ1009)

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

其中 $\begin{cases}\Delta u = \dfrac{\partial u}{\partial x}\Delta x + \dfrac{\partial u}{\partial y}\Delta y,\\ \Delta v = \dfrac{\partial v}{\partial x}\Delta x + \dfrac{\partial v}{\partial y}\Delta y.\end{cases}$

所以, $\Delta f = \Delta u + i\Delta v = \left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)\Delta x + \left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right)\Delta y$,

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

因为 $\begin{cases}\Delta x = \dfrac{1}{2}(\Delta z + \Delta \bar{z}),\\ \Delta y = -\dfrac{1}{2}i(\Delta z - \Delta \bar{z}),\end{cases}$

则 $$\begin{aligned}\Delta f &= \frac{\partial f}{\partial x}\left(\frac{\Delta z + \Delta \bar{z}}{2}\right) + \frac{\partial f}{\partial y}\left(-i\frac{\Delta z - \Delta \bar{z}}{2}\right)\\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x}\Delta z + \frac{\partial f}{\partial x}\Delta \bar{z} - i\frac{\partial f}{\partial y}\Delta z + i\frac{\partial f}{\partial y}\Delta \bar{z}\right)\\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\Delta z + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\Delta \bar{z}.\end{aligned}$$

又因为 $\Delta f = \dfrac{\partial f}{\partial z}\Delta z + \dfrac{\partial f}{\partial \bar{z}}\Delta \bar{z}$, 对比得:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

3.1　定理 1 的证明

由引理 $\dfrac{\partial f}{\partial \bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right) + \dfrac{1}{2}i\left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right) = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right) + \dfrac{1}{2}i\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right)$

若在区域 D 内有 $\dfrac{\partial f}{\partial \bar{z}} = 0$, 可推出 $\dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right) = 0$ 且 $\dfrac{1}{2}i\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) = 0$ 即 $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, 反之若在区域 D 内 $u(x,y)$ 和 $v(x,y)$ 满足 Cauchy-Riemann 方程即 $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, 则有 $\dfrac{\partial f}{\partial \bar{z}} = 0$.

3.2　定理 2 的证明

因为 $f(z)$ 在区域 D 内解析, 故 $f'(z)$, $f'(\bar{z})$ 也在区域 D 内解析, 满足 Cauchy-Riemann 方程即 $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\\ &= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right)\\ &= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right)\\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{\frac{\Delta z + \Delta \bar{z}}{2}}.\end{aligned}$$

取 $g(z) = \left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)\Big|_{\substack{x=z\\y=0}}$, 于是 $g(z)$ 在区域 D 内解析, 即 $z = x$ 取实数时, 由导数的唯一性知:

$$f'(z) = g(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Big|_{\substack{x=z\\y=0}},$$

$$\begin{aligned}f(z) &= \int\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{(z,0)}\mathrm{d}z\\ &= \int \frac{\partial u}{\partial x}_{(z,0)}\mathrm{d}z + i\int \frac{\partial v}{\partial x}_{(z,0)}\mathrm{d}z\\ &= u(z,0) + iv(z,0).\end{aligned}$$

即 $f(\bar{z}) = f(z) = u(z,0) + iv(z,0)$.

4　应　用

例 4.1　已知解析函数 $f(z)$ 实部为 $u(x,y) = x^3 - 3xy^2$, 求 $f(z)$.

解: 因为 $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\dfrac{\partial u}{\partial y} = -6xy$, 则

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}y + C$$

由线积分法可得:

$$\begin{aligned}v(x,y) &= \int_{(0,0)}^{(x,y)} 6xy\mathrm{d}x + (3x^2 - 3y^2)\,\mathrm{d}y + C\\ &= \int_{(0,0)}^{(x,0)} 6xy\mathrm{d}x + (3x^2 - 3y^2)\,\mathrm{d}y +\\ &\quad \int_{(x,0)}^{(x,y)} 6xy\mathrm{d}x + (3x^2 - 3y^2)\,\mathrm{d}y + C\\ &= \int_0^y (3x^2 - 3y^2)\,dy + C\\ &= 3x^2y - y^3 + C.\end{aligned}$$

故: $f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3 + C)$.

令 $x = z, y = 0$, 得

$$f(z) = (x + iy)^3 + iC = z^3 + iC.$$

例 4.2　已知解析函数 $f(z)$ 的实部为 $u(x,y) = x^3 + 6x^2y - 3xy^2 - 2y^3$, 且 $f(0) = 0$, 求 $f(z)$.

解: 因为 $\dfrac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2$, $\dfrac{\partial u}{\partial y} = 6x^2 - 6xy - 6y^2$,

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net

又因为函数 $f(z)$ 解析,满足 Cauchy-Riemann 方程,由全微分法,得:

$$\begin{aligned} dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= -(6x^2 - 6xy - 6y^2)\,dx + (3x^2 + 12xy - 3y^2)\,dy, \end{aligned}$$

则

$$\begin{aligned} v(x,y) &= \int -\frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy - k_1(x,y) + c \\ &= (3x^2y + 6xy^2 - 2x^3) + (3x^2y + 6xy^2 - y^3) \\ &\quad - (3x^2y + 6xy^2) + c \\ &= 3x^2y + 6xy^2 - 2x^3 - y^3 + c. \end{aligned}$$

故
$$\begin{aligned} f(z) &= (x^3 + 6x^2y - 3xy^2 - 2y^3) + i(3x^2y \\ &\quad + 6xy^2 - 2x^3 - y^3 + c) \\ &= (x^3 - 3xy^2) + i(3x^2 - y^3) \\ &\quad + 2[3x^2 - y^3) + i(3xy^2 - x^3)] + ic. \end{aligned}$$

令 $x = z, y = 0$,得 $f(z) = z^3 - 2iz^3 + ic$,因为 $f(0) = 0$,则 $c = 0$. 于是

$$f(\bar{z}) = \bar{z}^3 - 2i\bar{z} = (1 - 2i)\bar{z}.$$

注记:令 $x = z, y = 0$ 或指数函数均可. 由 $f(\bar{z}) = e^{i\bar{z}} = e^{y}(\cos x + i\sin x)$,则

$$f(x) = \cos x + i\sin x = e^{ix} = e^{iz}.$$

但反过来,$x = 0, iy = z$,此时 $f(\bar{z}) = u(0,\bar{z}) + iv(0,\bar{z})$ 则不满足.

参 考 文 献

[1] 钟玉泉. 复变函数论[M]. 3 版. 北京:高等教育出版社,2004.

[2] 王见定. 半解析函数与共轭解析函数[M]. 北京:北京工业大学出版社,1988.

[3] 仝泽柱,娄正凯. 复变函数共轭解析的充要条件[J]. 徐州工程学院学报,2006(3):97 - 100.

[4] 王海英. 复变函数共轭可微的又一充要条件及应用[J]. 吉林师范大学学报,2008(2):82 - 83.

[5] 秦静. 求解析函数的一个方法[J]. 山东师大学报(自然科学版),1996,11(4):99 - 100.

[6] 张子杰. 由调和函数求对应解析函数的简便方法[J]. 河北工程技术高等专科学校学报,2006(2):46 - 47.

[7] 王信松. 解析函数的一种简单求法[J]. 淮北煤炭师范学院学报,2006,27(2):26 - 27.

[8] 宋晋生. 解析函数的一种简易求法[J]. 长治学院学报,2006,23(2):84.

A Note on One Method of Conjugate Analytic Function

HUANG Wei

(Baoji Vocational and Technical College, Baoji Shaanxi 721013)

Abstract: The sufficient and necessary conditions, one methods and important applications of complex conjugate analytical functions are given.

Keywords: analytic function; conjugate analytic functions; sufficient and necessary conditions; method; important applications

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net