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关于欧拉—马克劳林求和公式的应用研究*

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摘 要

研究了欧拉—马克劳林求和公式,目的是推广欧拉—马克劳林求和公式的应用;采用了数论特殊函数和解析数论相结合的方法;通过欧拉—马克劳林求和公式给出了三个重要的结论,通过伯努利级数和欧拉常数表示了 $\sum_{k=1}^{n}\frac{1}{k}$,利用伯努利级数和伯努利多项式积分得出并证明了重要结论 $\psi(x)$ 和 $\zeta(u,a)$;这些结论对于数论特殊函数的研究具有重要作用.

关键词: 欧拉—马克劳林求和,赫维茨 ζ 函数,伯努利级数,对数导数.
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1 引言和主要结果

文献[1]和[2]提出了关于求和公式的若干问题,文献[3-5]给出了 $\sum_{u_1<n\leqslant u_2}^{n} f(n)$,文献[6-8]得到了公式 $L_u(x,a)=\sum_{0\leqslant u\leqslant x}(n+a)^u$,展示了欧拉—马克劳林求和公式的重要性和实用性;文献[9-12]给出了赫维茨 ζ 函数公式的一些结果. 本文陈述并证明欧拉—麦克劳林公式 $\sum_{m+1}^{n} f(x)$,应用欧拉—马克劳林公式得到三个重要结果;应用伯努利级数和欧拉—马克劳林常数 γ 表示了 $\sum_{k=1}^{n}\frac{1}{k}$;应用伯努利级数和伯努利多项式积分得到了 $\Psi(x)$ 和赫维茨 $\zeta(u,a)$ 的重要结果.

命题 1 设 $f(x)$ 具有连续导数,则

$$\sum_{m+1}^{n} f(x)=\int_m^n f(x)\,dx+\sum_{\ell=1}^{s}(-1)^{\ell}\frac{B_\ell}{\ell!}\{f^{(\ell-1)}(n)-f^{(\ell-1)}(m)\}+\frac{(-1)^{s-1}}{s!}\int_m^n B_s(x-[x])f^{(s)}(x)\,dx. \tag{1}$$

其中 $B_s(x)$ 是伯努利多项式,伯努利数 $B_\ell=B_\ell(0)$.

定理 1 设 n,p 是正整数,则

$$\sum_{k=1}^{n}\frac{1}{k}=\log n+r+\frac{1}{2n}-\sum_{k=1}^{p-1}\frac{2B_{2k}}{2kn^{2k}}-\frac{\theta B_{2m}}{2p}, \tag{2}$$

其中 $0<\theta<1$,B_{2k} 是伯努利数,γ 是欧拉常数.

定理 2 设 $z\in C-(-\infty,0]$,则

$$\psi(z)=\log z-\frac{1}{2^z}-\sum_{j=1}^{m}\frac{B_{2j}}{2j}\frac{1}{z^{2j}}+\int_0^\infty\frac{B_{2m}(x-[x])}{(x+z)^{2m-1}}dx, \tag{3}$$

$$\psi^{(n)}(z)-\psi^{(n-1)}(z)=\frac{1}{2\cdot n!\,z^{n+1}}+(2j)_n\sum_{j=1}^{m}\cdot\frac{B_{2j}}{2j}\frac{1}{z^{2j+n}}+(2m)_n(-1)^{n+1}\int_0^\infty\cdot\frac{B_{2m}(x-[x])}{(x+z)^{2m+n-1}}dx. \tag{4}$$

定理 3 如果 s,t,ℓ 是正整数,u,a 是复变量,则

$$\zeta(s,a)=\frac{1}{a^s}+\frac{1}{(s-1)a^{s-1}}-\sum_{\ell=1}^{r}\frac{B_\ell}{\ell!}(s)_{\ell-1}\frac{1}{a^{s+l-1}}-\frac{(s)_t}{t!}\int_0^\infty\overline{B_t}(x)\frac{1}{(x+a)^{s+t}}dx, \tag{5}$$

$$\zeta(-u,a)=a^u-\frac{1}{u+1}a^{u+1}-\sum_{r=1}^{\ell}\frac{(-1)^r}{r}\binom{u}{r-1}B_1a^{u-r+1}+$$

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$$(-1)^{\ell+1}\binom{u}{\ell}\int_0^{\infty}\overline{B_t}(t)\,(t+a)^{u-\ell}dt. \quad (6)$$

2 引理及证明

为了证明定理,首先引入下面两个引理,并给出证明.

引理 1 设 $B_s(x)$ 是伯努利多项式,对于任意 x 有

$$\tilde{B}_n(x) = \frac{1}{n!}B_n(x-[x]). \quad (7)$$

证明: 设

$$G(x,t) := \sum_0^{\infty}\overline{B}_n(x)t^n, \overline{B}_n(0) = 1. \quad (8)$$

当 $0<x<1$,

$$\frac{\partial G}{\partial x} = \sum_{n=1}^{\infty}\overline{B}'_n(x)t^n = \sum_{n=1}^{\infty}\overline{B}_{n-1}(x)t^n$$
$$= t\sum_{n=1}^{\infty}\overline{B}_n(x)t^n = tG,$$

得到

$$G(x,t) := A(t)e^{xt}. \quad (9)$$

将 $\int_0^1\overline{B}_n(x)dx = 0$ 代入(8)和(9),得到

$$A(t)\frac{e^t-1}{t} = 1 \text{ 或 } A(t) = \frac{t}{e^t-1}.$$

因此,

$$G(x,t) = \frac{te^{xt}}{e^t-1} = \sum_0^{\infty}B_n(x)\frac{t^n}{n!}, \quad (10)$$

又由,

$$\overline{B}_n(x) = \frac{B_n(x)}{n!}, \quad 0<x<1,$$

又由 $\overline{B}_n(x)$ 的周期性,可得

$$\overline{B}_n(x) = \frac{1}{n!}B_n(x-[x]).$$

引理 2. 设 f 有连续的导数,则

$$\sum_{m+1}^{n}f(x) = \int_m^n f(x)dx + \frac{1}{2}(f(n)-f(m)) + \sum_{j=1}^{m-1}b_{2j}\{f^{(2j-1)}(n)-f^{(2j-1)}(m)\} + \int_m^n(b_{2m}-b_{2m}(x))f^{(2m)}dx. \quad (11)$$

证明: 由命题 1

$$B_\ell = B_\ell(0)\quad\left(\frac{B_\ell(x-[x])}{\ell!} = b_\ell(x-[x])\right),$$

$$b_{2\ell}(u) = (-1)^{\ell-1}\sum_{n=1}^{\infty}\frac{2}{(2n\pi)^{2\ell}}\cos(2n\pi u),$$

$$b_{2\ell+1}(u) = (-1)^{(\ell-1)}\sum_{n=1}^{\infty}\frac{2}{(2n\pi)^{2\ell+1}}\sin(2n\pi u),$$
$$\ell = 1,2,\cdots,$$

$$b_{2\ell+1}(0) = 0, \quad \frac{B_\ell}{\ell!} = b_\ell.$$

当 u 是整数时,$b_{2\ell+1}(u) = 0$.

$$b_{2\ell}(u) = (-1)^{l-1}\frac{1}{(2\pi)^{2l}}\sum_{n=1}^{\infty}\frac{1}{n^{2l}}, B_1 = -\frac{1}{2}, \ell = 2i,$$

$$\sum_{m+1}^{n}f(x) = \int_m^n f(x)dx + \frac{1}{2}(f(n)-f(m)) + \sum_{j=1}^{m}\frac{B_{2j}}{(2j)!}\{f^{(2j-1)}(n)-f^{(2j-1)}(m)\} + \frac{(-1)^{2j-1}}{(2j)!}\int_m^n B_{2j}(x-[x])f^{(2j)}(x)dx$$
$$= \int_m^n f(x)dx + \frac{1}{2}(f(n)-f(m)) + \sum_{j=1}^{m}b_{2j}\{f^{(2j-1)}(n)-f^{(2j-1)}(m)\} - \int_m^n b_{2j}(x-[x])f^{(2j)}(x)dx,$$

$$u = x-[x]\quad\left(\frac{B_\ell(x-[x])}{\ell!} = b_{2j}(x-[x])\right)$$
$$= \int_m^n f(x)dx + \frac{1}{2}(f(n)-f(m)) + \sum_{j=1}^{m-1}b_{2j}\{f^{(2j-1)}(n)-f^{(2j-1)}(m)\} + b_{2m}\{f^{2m-1}(n)-f^{2m-1}(m)\} - \int_m^n b_{2m}(u)f^{2m}(x)dx$$
$$= \int_m^n f(x)dx + \frac{1}{2}(f(n)-f(m)) + \sum_{j=1}^{m-1}b_{2j}\{f^{(2j-1)}(n)-f^{(2j-1)}(m)\} + \int_1^n(b_{2m}-b_{2m}(x))f^{(2m)}(x)dx,$$

$$\sum_{m+1}^{n}f(x) = \int_m^n f(x)dx + \frac{1}{2}(f(n)-f(m)) + \sum_{j=1}^{m-1}b_{2j}\{f^{(2j-1)}(n)-f^{(2j-1)}(m)\} + \int_m^n(b_{2m}-b_{2m}(x))f^{(2m)}dx.$$

3 命题及定理的证明

命题 1: 设 $f(x)$ 具有连续导数,则

$$\sum_{m+1}^{n}f(x) = \int_m^n f(x)dx + \sum_{\ell=1}^{s}(-1)^{\ell}\frac{B_\ell}{\ell!}\{f^{(\ell-1)}(n)-$$

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$$f^{(\ell-1)}(m)\}+\frac{(-1)^{s-1}}{s!}\int_m^n B_s(x-[x])f^{(s)}(x)\,dx.$$

证明: 设 $f(x)$ 为定义在区间 $[n,n]$ 上的可微函数,m,n 是整数,设 $m\leqslant j<n$,j 是整数. 我们认为作为第一近似值,

$$\int_j^{j+1}f(x)\,dx\approx\frac{1}{2}(f(j)+f(j+1)). \tag{12}$$

又因为

$$\frac{1}{2}(f(j)+f(j+1))=\int_j^{j+1}\frac{d}{dx}\left[\left(x-j-\frac{1}{2}\right)f(x)\right]dx$$

$$=\int_j^{j+1}f(x)\,dx+\int_j^{j+1}\left(x-j-\frac{1}{2}\right)f'(x)\,dx. \tag{13}$$

令 $j=m,m+1,\cdots,n-1$,可知

$$\frac{1}{2}(f(m)+f(m+1))=\int_m^{m+1}f(x)\,dx+\int_m^{m+1}\left|x-m-\frac{1}{2}\right|f'(x)\,dx. \tag{14}$$

$$\frac{1}{2}(f(m+1)+f(m+2))=\int_{m+1}^{m+2}f(x)\,dx+\int_{m+1}^{m+2}\left(x-(m+1)-\frac{1}{2}\right)f'(x)\,dx. \tag{15}$$

$$\frac{1}{2}(f(n-1)+f(n))=\int_{n-1}^{n}f(x)\,dx+\int_{n-1}^{n}\left|x-(n-1)-\frac{1}{2}\right|f'(x)\,dx. \tag{16}$$

将(14),(15),…,(16)求和,可知

$$\sum_{m+1}^{n}f(k)=-\frac{1}{2}(f(m)+f(n))+\int_m^n f(x)\,dx+\int_m^n\left(x-[x]-\frac{1}{2}\right)f'(x)\,dx.$$

因为 $B_1=B_1(0)=-\frac{1}{2}$,$B_1(x)=x-[x]-\frac{1}{2}$,所以,

$$\sum_{m+1}^{n}f(k)=-\frac{1}{2}(f(m)+f(n))+\int_m^n f(x)\,dx+\int_m^n B_1(x-[x])f'(x)\,dx.$$

因为 $B_1(x-[x])=\overline{B_1}(x)$,$\overline{B_2}$ 是周期函数. 尤其是,$\overline{B_2}(0)=\overline{B_2}(1)=\cdots=\overline{B_2}(j)=\overline{B_2}(j+1)=\cdots$. 由分部积分法可知

$$\int_j^{j+1}\overline{B_1}(x)\,dx=\overline{B_2}(0)(f^2(j+1)-f(j))-\int_j^{j+1}\overline{B_2}(x)f'(x)\,dx,$$

当 j 由 m 取到 n 可知

$$\int_m^n\overline{B_1}(x)\,dx=\overline{B_2}(0)(f'(n)-f(m))-\int_m^n\overline{B_2}(x)f''(x)\,dx, \tag{17}$$

对(17)式作求导数和由部分积分可得 $\overline{B_n}(x)$,$\overline{B_n}'(x)=\overline{B_{n-1}}(x)$ $(0<x<1,n\geqslant 1)$ 和 $\int_0^1\overline{B_n}(x)\,dx=0$,$n\geqslant 1$. 由以上结果,得到

$$\sum_{m+1}^{n}f(x)=\int_m^n f(x)\,dx+\sum_{\ell=1}^{s}(-1)^{\ell}\overline{B_\ell}(0)\{f^{(\ell-1)}(n)-f^{(\ell-1)}(m)\}+(-1)^{s-1}\int_m^n\overline{B_x}(x)f^{(s)}(x)\,dx.$$

又由引理 1,可得(1).

定理 1. 设 n,p 是正整数,则

$$\sum_{k=1}^{n}\frac{1}{k}=\log n+r+\frac{1}{2n}-\sum_{k=1}^{p-1}\frac{2B_{2k}}{2kn^{2k}}-\frac{\theta B_{2m}}{2p}$$

其中 $0<\theta<1$,B_{2k} 是伯努利数,γ 是欧拉常数.

证明: $\sum_{k=1}^{n}\frac{1}{k}=1+\sum_{2}^{n}\frac{1}{k}$.

由(11)中,$f(x)=\frac{1}{x}$,$m=1$,

$$\sum_{k=2}^{n}\frac{1}{k}=\int_1^n f(x)\,dx+\frac{1}{2}\left(\frac{1}{n}-1\right)+\sum_{j=1}^{m-1}b_{2j}\{(-1)^{2j-1}(2j-1)!\frac{1}{n^{2j}}-(-1)^{2j-1}(2j-1)!\}+\int_1^n(b_{2m}-b_{2m}(x))(2m!)\,\frac{1}{x^{2m+1}}dx,$$

$$\sum_{k=1}^{n}\frac{1}{k}=\log n+\frac{1}{2}+\sum_{j=1}^{m-1}(2j-1)!\,b_{2j}+(2m)!\int_1^{\infty}\cdot(b_{2m}-b_{2m}(x))\,\frac{1}{x^{2m-1}}dx+\frac{1}{2n}-\sum_{j=1}^{m-1}\frac{(2j-1)}{n^{2j}}b_{2j}-(2m)!\int_n^{\infty}(b_{2m}-b_{2m}(x))\,\frac{1}{x^{2m-1}}$$

$$=\log n+k_m+\Omega_m(n),$$

$$\Omega_m(n)=\frac{1}{2n}-\sum_{j=1}^{m-1}\frac{(2j-1)}{n^{2j}}b_{2j}-(2m)!\int_n^{\infty}(b_{2m}-b_{2m}(x))\,\frac{1}{x^{2m-1}},$$

因为 $n\to\infty$,$\lim_{n\to\infty}\Omega_m(n)=0$,则有

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)=k_m=\gamma,$$

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$$\sum_{k=1}^{n}\frac{1}{k}=\log n+r+\frac{1}{2n}-\sum_{j=1}^{m-1}\frac{(2j-1)!}{n^{2j}}b_{2j}-(2m)!\int_n^{\infty}(b_{2m}-b_{2m}(x))\frac{1}{x^{2m+1}}dx,\quad(18)$$

$$2jb_{2j}=B_{2j},$$

现在我们考虑 $\int_n^{\infty}(b_{2m}-b_{2m}(x))\frac{1}{x^{2m+1}}dx$ 的积分,

由[1],知

$$b_{2l}=(-1)^{l-1}\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^{2l}},$$

$$b_{2l}(x)=(-1)^{l-1}\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^{2l}}\cos(2n\pi x),$$

$$b_{2l}-b_{2l}(x)=(-1)^{l-1}\sum_{n=1}^{\infty}\frac{1}{(2n\pi)^{2l}}(1-\cos(2n\pi x)),$$

$$b_{2l}-b_{2l}(x)=(-1)^{l-1}\sum_{n=1}^{\infty}\frac{\theta}{(2n\pi)^{2l}}=\theta b_{2l}.$$

当 $0<\theta<1$ 时,

$$\int_n^{\infty}(b_{2m}-b_{2m}(x))\frac{1}{x^{2m+1}}dx=\int_n^{\infty}\theta b_{2m}\frac{1}{x^{2m+1}}dx=\frac{\theta b_{2m}}{2mn^{2m}}.\quad(19)$$

将(19)代入(18),得到

$$\sum_{k=1}^{n}\frac{1}{k}=\log n+r+\frac{1}{2n}-\sum_{j=1}^{m-1}\frac{(2j-1)!}{n^{2j}}b_{2j}-(2m)!\frac{\theta b_{2m}}{2mn^{2m}}=\log n+r+\frac{1}{2n}-\sum_{j=1}^{m-1}\frac{B_{2j}}{2jn^{2j}}-\frac{\theta B_{2m}}{2mn^{2m}}.$$

定理 2 设 $z\in C-(-\infty,0]$,则

$$\psi(z)=\log z-\frac{1}{2z}-\sum_{j=1}^{m}\frac{B_{2j}}{2j}\frac{1}{z^{2j}}+\int_0^{\infty}\frac{B_{2m}(x-[x])}{(x+z)^{2m+1}}dx$$

$$\psi^{(n)}(z)-\psi^{(n-1)}(z)=\frac{1}{2\cdot n!z^{n+1}}+(2j)_n\sum_{j=1}^{m}\frac{B_{2j}}{2j}\frac{1}{z^{2j+n}}+(2m)_n(-1)^{n+1}\int_0^{\infty}\frac{B_{2m}(x-[x])}{(x+z)^{2m+n+1}}dx.$$

证明: $\psi(z)=(\log\Gamma(z))'=\frac{\Gamma'(z)}{\Gamma(z)}$

$\Gamma(z)=\lim_{n\to\infty}\prod_{l=1}^{n}\frac{l}{z+l-1}\left(\frac{l+1}{l}\right)^{z-1}$,则

$$\log\Gamma(z)=\lim_{n\to\infty}\prod_{l=1}^{n}\frac{l}{z+l-1}\left(\frac{l+1}{l}\right),\quad(20)$$

$$\log\Gamma(z)=\left[(z-1)\sum_{l=1}^{n}\log\frac{l+1}{l}-\sum_{l=1}^{n}\log\frac{z+l-1}{l}\right]=\lim_{n\to\infty}\left[(z-1)\log(n+1)-\sum_{l=1}^{n}\log\frac{z+l-1}{l}\right].\quad(21)$$

又由引理 1,令

$$f(x)=\log\frac{x+z-1}{x}=\log(x+z-1)-\log x,$$

得到

$$\sum_{l=1}^{n}\log\frac{z+l-1}{l}=\log z+\sum_{l=2}^{n}\log\frac{z+l-1}{l}=\log z+\int_1^n[\log(x+z-1)-\log x]dx+\sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(n+z-1)^{2j-1}}-\frac{1}{n^{2j-1}}-\frac{1}{z^{2j-1}}+1\right]+\frac{1}{2}[\log(n+z-1)-\log n-\log z]+\frac{1}{2m}\int_0^n B_{2m}(x-[x])\left[\frac{1}{(z+x-1)^{2m}}-\frac{1}{x^{2m}}\right]dx.$$

又由 $B_1=-\frac{1}{2}$,$B_{2j}=0$, $j\geqslant 1$,可知,

计算上面的积分,观察到涉及 n 的部分如下,

$$(n+z+1)\log(n+z-1)-n\log n+\frac{1}{2}\log\frac{n+z-1}{n}+\sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(n+z-1)^{2j-1}}-\frac{1}{n^{2j-1}}\right].$$

设 $n\to\infty$,(21)得到

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+1+\sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}\left(\frac{1}{z^{2j-1}}-1\right)-\frac{1}{2m}\int_1^{\infty}B_{2m}(x-[x])\left[\frac{1}{(x+z-1)^{2m}}-\frac{1}{x^{2m}}\right]dx.\quad(22)$$

又

$$\log n!-\left(n+\frac{1}{2}\right)\log n+n=1+\int_1^n\frac{B_1(x-[x])}{x}dx,$$

$$\xi'(0)=-1-\int_1^{\infty}\frac{x-[x]-\frac{1}{2}}{x}dx,$$

$$\lim_{n\to\infty}\left[\log\Gamma(z)-z-\frac{1}{2}\right)\log z+z\right]=\frac{1}{2}\log 2\pi,$$

设 $z\to\infty$,则(22)可整理为

$$1-\sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}+\frac{1}{2m}\int_1^{\infty}\frac{B_{2m}(x-[x])}{x^{2m}}dx=\frac{1}{2}\log 2\pi.$$

由(22)可得

$$\log\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\sum_{j=1}^{m}\frac{B_{2j}}{2j(2j-1)}\frac{1}{z^{(2j-1)}}+$$

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$$\frac{1}{2}\log 2\pi - \frac{1}{2m}\int_0^{\infty} \frac{B_{2m}(x - [x])}{(x+z)^{2m}}dx,$$

$$\psi(z) = (\log\Gamma(z))' = \log z - \frac{1}{2z} - \sum_{j=1}^{m} \frac{B_{2j}}{2j}\frac{1}{z^{2j}} + \int_0^{\infty} \frac{B_{2m}(x - [x])}{(x+z)^{2m-1}}dx,$$

$$\psi^{n}(z) = \psi^{m+1}(z) + (-1)^{m+1}\frac{1}{2n!\,2^{2m+1}} + (2j)_m \sum_{j=1}^{m} \frac{B_{2j}}{2j}\frac{1}{z^{2j+m}} + (2m)_n (-1)^{m+1} \cdot \int_0^{\infty} \frac{B_{2m}(x - [x])}{(x+z)^{2m+n-1}}dx,$$

$$\psi^{(n)}(z) - \psi^{(n-1)}(z) = (-1)^{n+1}\frac{1}{2\cdot n!\,z^{n+1}} + (2j)_n \sum_{j=1}^{m} \frac{B_{2j}}{2j}\frac{1}{z^{2j+n}} + (2m)_n (-1)^{n+1} \cdot \int_0^{\infty} \frac{B_{2m}(x - [x])}{(x+z)^{2m+n-1}}dx,$$

定理得证.

定理 3. 如果 s,t,ℓ 是正整数,u,a 是复变量,则

$$\zeta(s,a) = \frac{1}{a^s} + \frac{1}{(s-1)a^{s-1}} - \sum_{\ell=1}^{r} \frac{B_\ell}{\ell!}(s)_{\ell-1}\frac{1}{a^{s+l-1}} - \frac{(s)_t}{t!}\int_0^{\infty} \overline{B_t}(x)\ \frac{1}{(x+a)^{s+t}}dx,$$

$$\zeta(-u,a) = a^u - \frac{1}{u+1}a^{u+1} - \sum_{r=1}^{\ell} \frac{(-1)^r}{r}\binom{u}{r-1}B_1 a^{u-r+1} + (-1)^{\ell+1}\binom{u}{\ell}\int_0^{\infty} \overline{B_t}(t)\ (t+a)^{u-\ell}dt.$$

证明: $\zeta(s,a) = \sum_{n=o}^{\infty} \frac{1}{(n+a)^s} = a^{-s} + \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}$,

由命题 1 中的(1),设

$$f(x) = \frac{1}{(x+a)^s}, m = 0, n \to \infty,$$

$$f^{(l-1)}(x) = \frac{1}{(x+a)^s} = (-1)^{l-1}(s)_{l-1}\frac{1}{(x+a)^{s+l-1}},$$

$$f^{(t)}(x) = \frac{1}{(x+a)^s} = (-1)^t (s)_t \frac{1}{(x+a)^{s+t}},$$

$$\zeta(s,a) = a^{-s} + \frac{a^{1-s}}{s-1} + \sum_{\ell=1}^{r} \frac{(s)_{l-1}}{\ell!}B_\ell a^{-s-\ell+1} - \frac{(s)_t}{\ell!}\int_0^{\infty} \overline{B_t}(x)\ \frac{1}{(x+a)^{s+t}}dx.$$

当 $s \to -u$ 时,

$$\frac{(s)_{\ell-1}}{\ell!} = \frac{(-u)_{\ell-1}}{\ell!} = \frac{(-u)(-u+1)\cdots(-u+\ell-2)}{\ell!} = \frac{(-1)^{(\ell-1)}}{\ell!}u(u-1)\cdots(u-(\ell-1)+1) = \frac{(-1)^{(\ell-1)}}{\ell!}\frac{u(u-1)\cdots(u-(\ell-1)+1)}{(\ell-1)!} = \frac{(-1)^{(\ell-1)}}{\ell}\binom{\mu}{\ell-1},$$

$$\frac{(s)_t}{t!} = \frac{(-1)^{(\ell-1)}}{t!}(-u)_t = (-1)^{(\ell-1)}\binom{\mu}{t},$$

$$\zeta(-u,a) = a^u - \frac{a^{u+1}}{u+1} + 2\sum_{\ell=1}^{r} \frac{(-1)^{(\ell-1)}}{\ell}\binom{\mu}{\ell-1}B_\ell a^{u-l+1} - (-t)^t\binom{\mu}{t}\int_0^{\infty} \overline{B_\ell(x)}\,(x+a)^{(u-t)}dx = {}^{au} - \frac{a^{u+1}}{u+1} - \sum_{\ell=1}^{r} \frac{(-1)^\ell}{\ell}\binom{\mu}{\ell-1}B_\ell{}^{au-l+1} + (-1)^{l+1}\binom{\mu}{t}\int_0^{\infty} \overline{B_\ell(x)}\,(x+a)^{(u-t)}dx,$$

定理得证.

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4th Order Willmore Revolution Hypersurfaces in $\mathbf{R}^5$

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Abstract

In this paper, we establish the ordinary differential equation of 4th order revolution Willmore hypersurface in Euclidean space $\mathbf{R}^5$ and give a type of 4th order Willmore revolution hypersurface.

Key words: Conformal invariants, nth order Willmore functional, 4th order Willmore revolution hypersurface.

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The Euler-Maclaurin Summation Formula and It's Application

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Abstract

Euler-Maclaurin Summation Formula was been researched, the purpose was to promote the application of it. The innovation of method was combination with number theory special function and analytic number theory. Three important results were gotten by the Euler-Maclaurin Summation Formula, $\sum_{k=1}^{n}\frac{1}{k}$ was expressed by Bernoulli series and Euler-Maclaurin, $\Psi(x)$ and Hurwitz Zeta-function $\zeta(u,a)$ were stated and proved by Bernoulli series and integral of Bernoulli polynomials. It was very important to research the number theory special function.

Key words: Euler-maclaurin summation, hurwitz zeta-function, bernoulli series, logarithmic derivative.

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