**DOI**：10.19789/j.1004-9398.2022.01.002
**文献引用**：赵宇，康兆敏，刘琳，等.广义单调性与广义凸性[J].首都师范大学学报（自然科学版），2022，43（1）：7-11+40. ZHAO Y，KANG Z M，LIU L，et al. Generalized monotonicity and generalized convexity[J]. Journal of Capital Normal University（Natural Science Edition），2022，43（1）：7-11+40.

# 广义单调性与广义凸性*

赵　宇，康兆敏，刘　琳，刘春妍，黄金莹**
（佳木斯大学理学院，黑龙江 佳木斯　154007）

**摘要**：在广义单调性方面做进一步的推广，并建立函数的广义凸性与其梯度向量的广义单调性之间的等价关系.首先，建立$F$单调映射、$F$伪单调映射和$F$拟单调映射概念.其次，利用可微$F$凸（伪凸、拟凸）函数的梯度等价刻画，结合广义单调映射概念以及微分性质，研究广义凸性与广义单调性的内在联系：在一定条件下，$f$是$K$上的$F$凸函数，当且仅当$\nabla f$是$K$上的$F$单调映射；$f$是$K$上的$F$伪凸函数，当且仅当$\nabla f$是$K$上的$F$伪单调映射；$f$是$K$上的$F$拟凸函数，当且仅当$\nabla f$是$K$上的$F$拟单调映射.最后，给出某些具体广义凸函数的广义单调性刻画.

**关键词**：$F$单调映射；$F$伪单调映射；$F$拟单调映射；梯度
**中图分类号**：O174.13

# Generalized monotonicity and generalized convexity*

ZHAO Yu，KANG Zhaomin，LIU Lin，LIU Chunyan，HUANG Jinying**
（Department of Mathematics，Jiamusi University，Jiamusi Heilongjiang　154007）

**Abstract**：This article extends some conclusions of generalized monotonicity and establishes equivalent relationship between generalized convexity of functions and the generalized monotonicity of their gradient. Firstly，give the concepts of $F$-monotone maps，$F$-pseudo monotone maps and $F$-quasi monotone maps. Secondly，use gradient equivalent descriptions of differentiable $F$-convex（pseudo convex，quasi-convex）functions，and combine the concepts of generalized monotone maps and the properties of differentiation to study the relationship between generalized monotonicity and generalized convexity. The results show that under certain conditions，$f$ is $F$-convex functions over $K$，if and only if $\nabla f$ is $F$-monotone maps over $K$. $f$ is $F$- pseudo functions over $K$，if and only if $\nabla f$ is $F$- pseudo monotone maps over $K$. $f$ is $F$- quasi functions over $K$，if and only if $\nabla f$ is $F$- quasi monotone maps over $K$. Lastly，based on the application，give some generalized monotonicity descriptions of generalized convex functions.

**Keywords**：$F$-monotone maps；$F$-pseudo monotone maps；$F$-quasi monotone maps；gradient
**CLC**：O174.13

收稿日期：2021-01-04
* 佳木斯大学科学技术研究项目（L2014-016）
** 通信作者：hjyshuxue@163. com

## 0 引 言

通过研究函数凸性的梯度刻画,获得梯度向量映射的单调性,反过来可作为函数凸性的有力判据.随着函数凸性的不断推广以及相应的梯度刻画结论的获得,单调映射的概念也不断地被推广并加以研究,这不仅使得广义凸性的刻画更加丰富,同时也极大丰富了优化理论内容.1976年,Karamardian[1]提出伪单调性;1990年,Karamardian和Schaible[2]提出拟单调性,并给出了伪凸性与伪单调性、拟凸性与拟单调性之间的等价性.以此为开端,国内外学者依托广义凸性的研究,相应演绎出大量的广义单调性概念.2003年,Yang等[3]建立预不变凸函数与一些广义单调性之间的关系;2004年,Singh和Pini[4]建立了$G$-广义凸性与$G$-广义单调映射之间的关系;2007年,彭再云等[5]建立了$G$-伪凸性与$G$-伪单调性;2014年,陈乔和罗杰[6]提出了$E$-伪单调性和$E$-拟单调性;2016年,杨新民和戎卫东[7]对广义单调性的基础性工作做了较为全面论述.近几年来,广义凸性又有了新的推广.2017年,黄金莹等[8]和赵宇等[9]通过在点集上建立广义凸结构的方法,给出并研究了$F$凸函数、$F$拟凸函数.

本文将在以上文献的基础上,结合文献[8-9]的广义凸结构理论,建立相应的$F$单调映射、$F$伪单调映射和$F$拟单调映射概念,进一步研究广义凸性与广义单调性之间的关系.

## 1 单调映射概念

**定义1** 设非空集合$K\subseteq\mathbf{R}^n$,向量值函数$F:K\times K\times[0,1]\to\mathbf{R}^n$.称$F$是$K$上的广义凸结构,如果$F$满足以下4个条件:

(1) $\forall\lambda\in[0,1],\forall x\in K$ 有 $F(x,x,\lambda)=x$(单点性);

(2) $\forall x,y\in K$,有 $F(x,y,0)\in K,F(x,y,1)\in K$(端点性);

(3)对 $\forall\alpha,\beta\in[0,1]$,且$\alpha<\beta$,$\forall x,y\in K$,有:当$F(x,y,\beta)\in K$时,$F\left[y,F(x,y,\beta),1-\frac{\alpha}{\beta}\right]=F(x,y,\alpha)$,(条件$P_1$);当$F(x,y,\alpha)\in K$时,$F\left[x,F(x,y,\alpha),\frac{\beta-\alpha}{1-\alpha}\right]=F(x,y,\beta)$,(条件$P_2$);

(4)$F$关于$\lambda$在$[0,1]$上连续($\lambda$连续性).

**定义2** 设$F$是$K\subseteq\mathbf{R}^n$上的广义凸结构,且$F$关于$\lambda$在$[0,1]$上可微,$\nabla:K\to\mathbf{R}^n$为映射.

(1)称$\nabla$是$K$上的$F$单调映射,如果对$\forall x,y\in K$有$F'_\lambda(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla(y)+F'_\lambda(y,x,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla(x)\leqslant 0$; (1)

(2)称$\nabla$是$K$上的$F$伪单调映射,如果对$x,y\in K$且$F'_\lambda(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla(y)\geqslant 0$时,有$F'_\lambda(y,x,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla(x)\leqslant 0$; (2)

(3)称$\nabla$是$K$上的$F$拟单调映射,如果对$x,y\in K$且$F'_\lambda(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla(y)>0$时,有$F'_\lambda(y,x,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla(x)\leqslant 0$. (3)

**注1** 约定符号"$F'_\lambda(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}$"的含义是"$\left[F'_\lambda(x,y,\lambda)\big|_{\lambda=0}\right]^{\mathrm{T}}$",T表示转置.从定义2可以直接得到,$F$单调映射一定是$F$伪单调映射,$F$伪单调映射一定是$F$拟单调映射.

**注2** 在定义2中,特取$F(x,y,\lambda)=\lambda x+(1-\lambda)y$,$\forall x,y\in K,\lambda\in[0,1]$,则

$$F'_\lambda(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}=(x-y)^{\mathrm{T}},F'_\lambda(y,x,\lambda)\big|_{\lambda=0}^{\mathrm{T}}=(y-x)^{\mathrm{T}}.$$

代入式(1)中得,

对$\forall x,y\in K$有$(x-y)^{\mathrm{T}}(\nabla(x)-\nabla(y))\geqslant 0$; (4)

代入式(2)中得,

对$\forall x,y\in K$且$(x-y)^{\mathrm{T}}\nabla(y)\geqslant 0$时,有

$$(x-y)^{\mathrm{T}}\nabla(x)\geqslant 0; \tag{5}$$

代入式(3)中得,

对$x,y\in K$且$(x-y)^{\mathrm{T}}\nabla(y)>0$时,有

$$(x-y)^{\mathrm{T}}\nabla(x)\geqslant 0. \tag{6}$$

式(4~6)分别是单调映射、伪单调映射和拟单调映射的具体定义.

**定义3** 设非空集合$K\subseteq\mathbf{R}^n$,向量值函数$F:K\times K\times[0,1]\to\mathbf{R}^n$是$K$上的广义凸结构,称$K$是关于$F$的凸集,如果

$$\forall\lambda\in(0,1),\forall x,y\in K\text{有}F(x,y,\lambda)\in K.$$

**定义4** 设$K\subseteq\mathbf{R}^n$是关于$F$的凸集,实值函数$f:K\to\mathbf{R}$满足:$\forall x,y\in K$有$F(x,y,0)=y$,称$f$是$K$上的$F$伪凸函数,如果对$\forall x,y\in K$且$f(y)>f(x)$,$\exists\alpha,\tau>0$,当$\lambda\in[0,\tau]$时,有

$$f[F(x,y,\lambda)]\leqslant f(y)-\lambda\alpha.$$

**引理1** 设$K\subseteq\mathbf{R}^n$是关于$F$的开凸集,实值函数$f:K\to\mathbf{R}$满足:$\forall x,y\in K$有$F(x,y,0)=y$,$f[F(x,y,1)]\leqslant f(x)$且$F$关于$\lambda$在$[0,1]$上可微,$f:K\to\mathbf{R}$在$K$上可微,$\nabla f:K\to\mathbf{R}^n$为$f$的梯度向

量,则:

(1)$f$是$K$上的$F$凸函数,当且仅当对$\forall x,y\in K$有$f(x)\geqslant f(y)+F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)$;

(2)$f$是$K$上的$F$伪凸函数,当且仅当对$x,y\in K$且$F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)\geqslant 0$时,有$f(x)\geqslant f(y)$;

(3)$f$是$K$上的$F$拟凸函数,当且仅当对$x,y\in K$且$f(x)\leqslant f(y)$时,有$F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)\leqslant 0$.

**证明**　仅证(2).首先证明必要性.

设$x,y\in K$且$f(y)>f(x)$,由定义4知,$\exists\alpha,\tau>0$,当$\lambda\in[0,\tau]$时,有$f[F(x,y,\lambda)]\leqslant f(y)-\lambda\alpha$.

由$F(x,y,0)=y$,可将上式变形为$\dfrac{f[F(x,y,\lambda)]-f[F(x,y,0)]}{\lambda}\leqslant-\alpha<0$,从而

$$F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)=\lim_{\lambda\to 0^{+}}\frac{f[F(x,y,\lambda)]-f[F(x,y,0)]}{\lambda}\leqslant-\alpha<0.$$

下面证明充分性.由充分性条件可知,当$x,y\in K$且$f(y)>f(x)$时,有$F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)<0$,

取定$\alpha(>0)$使之满足:$F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)<-\alpha<0$,即有

$$\lim_{\lambda\to 0^{+}}\frac{f[F(x,y,\lambda)]-f[F(x,y,0)]}{\lambda}<-\alpha<0.$$

从而$\exists\tau\in(0,1]$,使得当$\lambda\in[0,\tau]$时,有$f[F(x,y,\lambda)]-f[F(x,y,0)]<-\lambda\alpha$,即

$$f[F(x,y,\lambda)]\leqslant f(y)-\lambda\alpha.$$

**注3**　对于引理1结论(3),可将前提条件$f[F(x,y,1)]\leqslant f(x)$改为$f[F(x,y,1)]\leqslant\max\{f(x),f(y)\}$.

**引理2**　设$F$是$K$上的广义凸结构,且$F(x,y,u_1)\in K,F(x,y,u_2)\in K$,其中$x,y\in K,u_1,u_2\in[0,1]$,$F$关于$\lambda$在$[0,1]$上可微,则

$$F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=u_2}(u_1-u_2)=F'_{\lambda}\left[F(x,y,u_1),F(x,y,u_2),\lambda\right]\big|^{\mathrm{T}}_{\lambda=0}.$$

## 2　广义凸性与广义单调性之间的关系

**定理1**　设$K\subseteq\mathbf{R}^n$是关于$F$的开凸集,实值函数$f:K\to\mathbf{R}$满足:$\forall x,y\in K$有$F(x,y,0)=y$,$f[F(x,y,1)]\leqslant f(x)$且$F$关于$\lambda$在$[0,1]$上可微,$f:K\to\mathbf{R}$在$K$上可微,则$f$是$K$上的$F$凸函数,当且仅当$\nabla f$是$K$上的$F$单调映射.

**证明**　必要性.由引理1结论(1)知,当$f$是$K$上的$F$凸函数时,对$\forall x,y\in K$有

$$f(x)\geqslant f(y)+F'_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y),$$

$$f(y)\geqslant f(x)+F'_{\lambda}(y,x,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(x).$$

两式相加即得

$$F''_{\lambda}(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)+F'_{\lambda}(y,x,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(x)\leqslant 0.$$

所以,$\nabla f$是$K$上的$F$单调映射.下面利用反证法证充分性.

设$\nabla f$是$K$上的$F$单调映射,但$f$不是$K$上的$F$凸函数,则

$$\exists\lambda_0\in(0,1),\exists x_0,y_0\in K,\text{有}f\left[F\left(x_0,y_0,\lambda_0\right)\right]>\lambda_0 f\left(x_0\right)+\left(1-\lambda_0\right)f\left(y_0\right),\tag{7}$$

注意到$F\left(x_0,y_0,0\right)=y_0$,$f\left[F\left(x_0,y_0,1\right)\right]\leqslant f\left(x_0\right)$,就有

$$\lambda_0 f\left(x_0\right)+\left(1-\lambda_0\right)f\left(y_0\right)\geqslant\lambda_0 f\left[F\left(x_0,y_0,1\right)\right]+\left(1-\lambda_0\right)f\left[F\left(x_0,y_0,0\right)\right].\tag{8}$$

由式(7~8)得

$$f\left[F\left(x_0,y_0,\lambda_0\right)\right]>\lambda_0 f\left[F\left(x_0,y_0,1\right)\right]+\left(1-\lambda_0\right)f\left[F\left(x_0,y_0,0\right)\right].\tag{9}$$

将式(9)改写

$$\lambda_0\left\{f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,1\right)\right]\right\}+\left(1-\lambda_0\right)\left\{f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,0\right)\right]\right\}>0.\tag{10}$$

由中值定理知,$\exists\lambda_1\in\left(0,\lambda_0\right),\lambda_2\in\left(\lambda_0,1\right)$,使得

$$f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,1\right)\right]=\left(\lambda_0-1\right)F'_{\lambda}\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_2}\nabla f\left[F\left(x_0,y_0,\lambda_2\right)\right],$$

$$f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,0\right)\right]=\lambda_0 F'_{\lambda}\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_1}\nabla f\left[F\left(x_0,y_0,\lambda_1\right)\right].$$

由引理2计算可得

$$\lambda_0 F'_{\lambda}\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_1}\nabla f\left[F\left(x_0,y_0,\lambda_1\right)\right]=\frac{\lambda_0}{\lambda_2-\lambda_1}F'_{\lambda}\left[F\left(x_0,y_0,\lambda_2\right),F\left(x_0,y_0,\lambda_1\right)\right]\big|^{\mathrm{T}}_{\lambda=0}\nabla f\left[F\left(x_0,y_0,\lambda_1\right)\right],\tag{11}$$

$$\left(\lambda_0-1\right)F'_{\lambda}\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_2}\nabla f\left[F\left(x_0,y_0,\lambda_2\right)\right]=\frac{\lambda_0-1}{\lambda_1-\lambda_2}F'_{\lambda}\left[F\left(x_0,y_0,\lambda_1\right),F\left(x_0,y_0,\lambda_2\right)\right]\big|^{\mathrm{T}}_{\lambda=0}\nabla f\left[F\left(x_0,y_0,\lambda_2\right)\right].\tag{12}$$

将式(11~12)代入式(10),得

$$\frac{\lambda_0(\lambda_0-1)}{\lambda_1-\lambda_2}F_\lambda'\left[F(x_0,y_0,\lambda_1),F(x_0,y_0,\lambda_2)\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_2)\right]+$$

$$\frac{(1-\lambda_0)\lambda_0}{\lambda_2-\lambda_1}F_\lambda'\left[F(x_0,y_0,\lambda_2),F(x_0,y_0,\lambda_1)\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_1)\right]>0,$$

即

$$F_\lambda'\left[F(x_0,y_0,\lambda_1),F(x_0,y_0,\lambda_2)\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_2)\right]+F_\lambda'\left[F(x_0,y_0,\lambda_2),F(x_0,y_0,\lambda_1)\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_1)\right]>0. \tag{13}$$

与$\nabla f$是$K$上的$F$单调映射矛盾.

**定理2** 设$K\subseteq\mathbf{R}^n$是关于$F$的开凸集,实值函数$f:K\to\mathbf{R}$满足:$\forall x,y\in K$有$F(x,y,0)=y$,$f[F(x,y,1)]\leqslant f(x)$且$F$关于$\lambda$在$[0,1]$上可微,$f:K\to\mathbf{R}$在$K$上可微,则$f$是$K$上的$F$伪凸函数,当且仅当$\nabla f$是$K$上的$F$伪单调映射.

**证明** 必要性.由引理1结论(2)知,当$f$是$K$上的$F$伪凸函数时,

对$x,y\in K$,$F_\lambda'(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(y)\geqslant 0$时,有

$$f(x)\geqslant f(y).$$

$F$伪凸函数是$F$拟凸函数,由引理1结论(3)进一步就有

当$f(x)\geqslant f(y)$时,$F_\lambda'(y,x,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(x)\leqslant 0$,

即:对$x,y\in K$且$F_\lambda'(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(y)\geqslant 0$时,有$F_\lambda'(y,x,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(x)\leqslant 0$.所以,$\nabla f$是$K$上的$F$伪单调映射.

反证法证明充分性.设$\nabla f$是$K$上的$F$伪单调映射,但$f$不是$K$上的$F$伪凸函数,即$\exists x_0,y_0\in K$虽然$F_\lambda'(x_0,y_0,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(y_0)\geqslant 0$时,但$f(x_0)<f(y_0)$.注意到$F(x_0,y_0,0)=y_0$,$f\left[F(x_0,y_0,1)\right]\leqslant f(x_0)$,就有$f\left[F(x_0,y_0,1)\right]<f\left[F(x_0,y_0,0)\right]$.由中值定理,$\exists\lambda_0\in(0,1)$,使

$$0<f\left[F(x_0,y_0,1)\right]-f\left[F(x_0,y_0,0)\right]=F_\lambda'(x_0,y_0,\lambda)\big|_{\lambda=\lambda_0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_0)\right]. \tag{14}$$

一方面,$F_\lambda'(x_0,y_0,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(y_0)\geqslant 0$,且由引理2可计算得

$$F_\lambda'(x_0,y_0,\lambda)\big|_{\lambda=0}^{\mathrm{T}}=\frac{1}{\lambda_0}F_\lambda'\left[F(x_0,y_0,\lambda_0),F(x_0,y_0,0),\lambda\right]\Big|_{\lambda=0}^{\mathrm{T}}$$

则

$$F_\lambda'(x_0,y_0,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(y_0)=\frac{1}{\lambda_0}\cdot F_\lambda'\left[F(x_0,y_0,\lambda_0),F(x_0,y_0,0),\lambda\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,0)\right]\geqslant 0.$$

即

$$F_\lambda'\left[F(x_0,y_0,\lambda_0),F(x_0,y_0,0),\lambda\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,0)\right]\geqslant 0. \tag{15}$$

对$F(x_0,y_0,0)$,$F(x_0,y_0,\lambda_0)$,由$F$伪单调性知,

$$F_\lambda'\left[F(x_0,y_0,0),F(x_0,y_0,\lambda_0),\lambda\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_0)\right]\leqslant 0. \tag{16}$$

另一方面,由引理2可计算得,

$$F_\lambda'(x_0,y_0,\lambda)\big|_{\lambda=\lambda_0}^{\mathrm{T}}=-\frac{1}{\lambda_0}F_\lambda'\left[F(x_0,y_0,0),F(x_0,y_0,\lambda_0),\lambda\right]\Big|_{\lambda=0}^{\mathrm{T}}.$$

将上式代入到式(14)中得,

$$F_\lambda'(x_0,y_0,\lambda)\big|_{\lambda=\lambda_0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_0)\right]=-\frac{1}{\lambda_0}\cdot F_\lambda'\left[F(x_0,y_0,0),F(x_0,y_0,\lambda_0),\lambda\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_0)\right]<0,$$

即

$$F_\lambda'\left[F(x_0,y_0,0),F(x_0,y_0,\lambda_0),\lambda\right]\Big|_{\lambda=0}^{\mathrm{T}}\nabla f\left[F(x_0,y_0,\lambda_0)\right]>0. \tag{17}$$

式(16)与(17)矛盾.

**定理3** 设$K\subseteq\mathbf{R}^n$是关于$F$的开凸集,实值函数$f:K\to\mathbf{R}$满足:$\forall x,y\in K$有$F(x,y,0)=y$,$f[F(x,y,1)]\leqslant\max\{f(x),f(y)\}$且$F$关于$\lambda$在$[0,1]$上可微,$f:K\to\mathbf{R}$在$K$上可微,则$f$是$K$上的$F$拟凸函数当且仅当$\nabla f$是$K$上的$F$拟单调映射.

**证明** 必要性.设$x,y\in K$且$F_\lambda'(x,y,\lambda)\big|_{\lambda=0}^{\mathrm{T}}\nabla f(y)>$

0,由$F$拟凸性及引理1结论(3)知,$f(x)>f(y)$,进而再由引理1结论(3)就有$F'_\lambda(y,x,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(x)\leqslant 0$. 即当

$$x,y\in K,\text{ 且 }F'_\lambda(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)>0\text{时},$$

$$\text{有 }F'_\lambda(y,x,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(x)\leqslant 0,$$

故$\nabla f$是$K$上的$F$拟单调映射.

充分性. 设$\nabla f$是$K$上的$F$拟单调映射,但$f$不是$K$上的$F$拟凸函数,则

$$\exists\lambda_0\in(0,1),\exists x_0,y_0\in K\text{有}f\left[F\left(x_0,y_0,\lambda_0\right)\right]>\max\left\{f\left(x_0\right)f\left(y_0\right)\right\}.$$

当$f\left(x_0\right)\geqslant f\left(y_0\right)$时,

$$0<f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left(y_0\right)=f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,0\right)\right],$$

$$0<f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left(x_0\right)\leqslant f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,1\right)\right].$$

当$f\left(x_0\right)<f\left(y_0\right)$时,

$$0<f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left(y_0\right)=f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,0\right)\right],$$

$$0<f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left(y_0\right)\leqslant f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,1\right)\right].$$

总有

$$f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,0\right)\right]>0,$$

$$f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,1\right)\right]>0. \quad (18)$$

由中值定理知,$\exists\lambda_1\in\left(0,\lambda_0\right),\lambda_2\in\left(\lambda_0,1\right)$,使得

$$f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,0\right)\right]=\lambda_0F'_\lambda\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_1}\nabla f\left[F\left(x_0,y_0,\lambda_1\right)\right], \quad (19)$$

$$f\left[F\left(x_0,y_0,\lambda_0\right)\right]-f\left[F\left(x_0,y_0,1\right)\right]=\left(\lambda_0-1\right)F'_\lambda\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_2}\nabla f\left[F\left(x_0,y_0,\lambda_2\right)\right]. \quad (20)$$

由引理2计算得

$$\lambda_0F'_\lambda\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_1}\nabla f\left[F\left(x_0,y_0,\lambda_1\right)\right]=\frac{\lambda_0}{\lambda_2-\lambda_1}F'_\lambda\left[F\left(x_0,y_0,\lambda_2\right),F\left(x_0,y_0,\lambda_1\right)\right]\big|^{\mathrm{T}}_{\lambda=0}\nabla f\left[F\left(x_0,y_0,\lambda_1\right)\right], \quad (21)$$

$$\left(\lambda_0-1\right)F'_\lambda\left(x_0,y_0,\lambda\right)\big|^{\mathrm{T}}_{\lambda=\lambda_2}\nabla f\left[F\left(x_0,y_0,\lambda_2\right)\right]=\frac{\lambda_0-1}{\lambda_1-\lambda_2}F'_\lambda\left[F\left(x_0,y_0,\lambda_1\right),F\left(x_0,y_0,\lambda_2\right)\right]\big|^{\mathrm{T}}_{\lambda=0}\nabla f\left[F\left(x_0,y_0,\lambda_2\right)\right]. \quad (22)$$

结合式(18~22)有

$$F'_\lambda\left[F\left(x_0,y_0,\lambda_2\right),F\left(x_0,y_0,\lambda_1\right)\right]\big|^{\mathrm{T}}_{\lambda=0}\nabla f\left[F\left(x_0,y_0,\lambda_1\right)\right]>0, \quad (23)$$

$$F'_\lambda\left[F\left(x_0,y_0,\lambda_1\right),F\left(x_0,y_0,\lambda_2\right)\right]\big|^{\mathrm{T}}_{\lambda=0}\nabla f\left[F\left(x_0,y_0,\lambda_2\right)\right]>0, \quad (24)$$

与$\nabla f$是$K$上的$F$拟单调映射矛盾.

## 3　应用举例

下面以定理1为例获得一些具体广义凸函数的广义单调性刻画.

定理1指出,在一定条件下,$f$是$K$上的$F$凸函数,当且仅当$\nabla f$是$K$上的$F$单调映射,即

$$\text{对}\forall x,y\in K\text{有}F'_\lambda(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(y)+F'_\lambda(y,x,\lambda)\big|^{\mathrm{T}}_{\lambda=0}\nabla f(x)\leqslant 0. \quad (25)$$

由于预不变凸函数、GA-凸函数分别是特取$F(x,y,\lambda)=y+\lambda\eta(x,y)$、$F(x,y,\lambda)=x^\lambda y^{1-\lambda}$的$F$凸函数,相应可求得$F''_\lambda(x,y,\lambda)\big|^{\mathrm{T}}_{\lambda=0}$的形式分别为$\eta(x,y)^{\mathrm{T}}$、$[y(\ln x-\ln y)]^{\mathrm{T}}$. $F''_\lambda(y,x,\lambda)\big|^{\mathrm{T}}_{\lambda=0}$的相应形式只需将$x,y$互换. 分别代入式(25),可得到一些具体可微广义凸函数的广义单调性刻画,例如关于预不变凸函数的结论[3]:

**例1**　设$K\subseteq\mathbf{R}^n$是开的关于$\eta(x,y)$不变凸集,且$\eta(x,y)$满足条件$C$,可微实值函数$f:K\to\mathbf{R}$满足:$f(y+\eta(x,y))\leqslant f(x),\forall x,y\in K$且$F$关于$\lambda$在$[0,1]$上可微,则$f$是$K$上的预不变凸函数当且仅当

$$\text{对}\forall x,y\in K\text{有}\eta(x,y)^{\mathrm{T}}\nabla f(y)+\eta(y,x)^{\mathrm{T}}\nabla f(x)\leqslant 0. \quad (26)$$

关于GA-凸函数的结论:

**例2**　设$K\subseteq\mathbf{R}^n_{++}$是开的几何凸集,$f:K\to\mathbf{R}$可微,则$f$为$K$上的GA-凸函数,当且仅当对$\forall x,y\in K$有

$$(\ln x-\ln y)^{\mathrm{T}}[y\nabla f(y)-x\nabla f(x)]\leqslant 0. \quad (27)$$

(下转第40页)

[13] SUBBU K P, GOZICK B, DANTU R. Locate me:magnetic-fields-based indoor localization using smartphones[J]. ACM Transactions on Intelligent Systems and Technology, 2013,4(4):73.
[14] HE S N, CHAN S H G. Wi-Fi fingerprint-based indoor positioning: recent advances and comparisons[J]. IEEE Communications Surveys and Tutorials, 2015, 18(1): 466-490.
[15] 王丽园,吴沐阳,吴家皋.基于WIFI BSSID相似度和RSSI概率分布的定位算法[J].计算机技术与发展,2016,26(12):200-204.
[16] YIU S, DASHTI M, CLAUSSEN H, et al. Wireless RSSI fingerprinting localization[J]. Signal Processing, 2017, 131:235-244.
[17] JAGADEESAN N A. Data-driven optimization for indoor localization[D]. Los Angeles: University of Southern California, 2018.
[18] HOU L J, LUO Y F, LIU Y H. Research on a fast matching method of K nearest neighbor for WiFi fingerprint location[C]//JIA Y M, DU J P, ZHANG W C. Proceedings of 2018 Chinese Intelligent Systems Conference. Wenzhou: Springer, 2019:555-565.
[19] LÄNGKVIST M, KARLSSON L, LOUTFI A. A review of unsupervised feature learning and deep learning for time-series modeling[J]. Pattern Recognition Letters, 2014,42:11-24.
[20] KHATAB Z E, HAJIHOSEINI A, GHORASHI S A. A fingerprint method for indoor localization using autoencoder based deep extreme learning machine[J]. IEEE Sensors Letters, 2018, 2(1):1-4.
[21] LUKIC Y, VOGT C, DÜRR O, et al. Speaker identification and clustering using convolutional neural networks[C]//BIE T D, LIJFFIJT J, MESNAGE C, et al. 2016 IEEE 26th international workshop on machine learning for signal processing (MLSP). Salerno: IEEE, 2016:1-6.

(责任编辑:兰丽丽)

(上接第11页)

特别地,设区间$(a,b)\subseteq(0,+\infty)$,$f$为定义在$(a,b)$上的可微函数,则$f$为$(a,b)$上的GA-凸函数,当且仅当$xf'(x)$为$(a,b)$上的递增函数.这是因为式(27)的一维形式是:

$$\text{对}\ \forall x,y\in(a,b)\subseteq(0,+\infty)\text{有}$$
$$(\ln x-\ln y)\left[yf'(y)-xf'(x)\right]\leqslant 0,$$

这就意味着$xf'(x)$为$(a,b)$上的递增函数.

## 参 考 文 献

[1] KARAMARDIAN S. Complementarity problems over cones with monotone and pseudo-monotone maps[J]. Journal of Optimization Theory and Applications, 1976, 18(4):445-454.
[2] KARAMARDIAN S, SCHAIBLE S. Seven kinds of monotone maps[J]. Journal of Optimization Theory and Applications, 1990, 66:37-46.
[3] YANG X M, YANG X Q, TEO K L. Generalized invexity and generalized invariant monotonicity[J]. Journal of Optimization Theory and Applications, 2003, 117(3):607-625.
[4] SINGH C, PINI R. G-monotonicity and G-convexity[J]. Journal of Information and Optimization Sciences, 2004, 25(2):287-301.
[5] 彭再云,黄应全,唐平.G-伪单调性、G-伪凸性及其应用[J].哈尔滨商业大学学报(自然科学版),2007,23(5):618-620.
[6] 陈乔,罗杰.E-广义单调性及其在广义变分不等式中的应用[J].重庆师范大学学报(自然科学版),2014,31(5):31-34.
[7] 杨新民,戎卫东.广义凸性及其应用[M].北京:科学出版社,2016.
[8] 黄金莹,张效菲,赵宇.广义凸集的结构理论[J].重庆师范大学学报(自然科学版),2016,33(1):56-62.
[9] 赵宇,黄金莹,康兆敏,等.广义凸函数的结构理论[J].重庆师范大学学报(自然科学版),2017,34(2):70-75.

(责任编辑:马田田)