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含余项的常微分方程数值计算方法

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摘 要

本文提出了一种新的微分方程计算方法的构建思路,并构建了一些公式,分析了算法特性,并给出了计算实例. 本文提出的算法具有稳定性好,精算精度高的特点,是一类具有使用价值的新方法.

关键词: 含余项的常微分方程计算方法,常微分方程,余项.

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1 一阶偏差分的表达形式

以往的微分方程计算方法都会忽略余项,这无疑会影响算法的精度,本文提出了包含余项的微分方程计算方法,并构建了一些公式. 由于余项可以使用一阶偏差分来表示,所以下面本文给出一阶偏差分的表达式.

设存在函数 $f(x,y)$,为了方便 $f(x_i,y_i)$ 也写成 f_i,符号 $\partial_x f(x_i,y_i)=\partial_x f_i$, $\partial_y f(x_i,y_i)=\partial_y f_i$ 分别表示 $f(x,y)$ 在点 (x_i,y_i) 处的一阶偏差分,设存在常数 $r>0$,$\partial_x f(x_i,y_i)$,$\partial_y f(x_i,y_i)$ 的具体表达式为:

$$\partial_x^{(+r)} f(x_i,y_i)=f(x_{i+r},y_i)-f(x_i,y_i) \tag{1.1}$$

$$\partial_x^{(-r)} f(x_i,y_i)=f(x_i,y_i)-f(x_{i-r},y_i) \tag{1.2}$$

$$\partial_y^{(+r)} f(x_i,y_i)=f(x_i,y_{i+r})-f(x_i,y_i) \tag{1.3}$$

$$\partial_y^{(-r)} f(x_i,y_i)=f(x_i,y_i)-f(x_i,y_{i-r}) \tag{1.4}$$

(1.1),(1.3)是向前偏差分,(1.2),(1.4)是向后偏差分. 当 $r=1$ 时称为一阶节点偏差分表达式,反之称为一阶非节点偏差分表达式. 下面把 $\partial_x^{(+r)} f(x_i,y_i)$,$\partial_x^{(-r)} f(x_i,y_i)$ 统一写成 $\partial_x^{(r)} f(x_i,y_i)$,或者 $\partial_x^{(r)} f_i$,$\partial_y^{(+r)} f(x_i,y_i)$,$\partial_y^{(-r)} f(x_i,y_i)$ 统一写成 $\partial_y^{(r)} f(x_i,y_i)$. 或者 $\partial_y^{(r)} f_i$,其中当 $r=1$ 时,$\partial_x^{(+1)} f(x_i,y_i)$,$\partial_x^{(-1)} f(x_i,y_i)$ 统一写成 $\partial_x^{(+)} f(x_i,y_i)$,$\partial_x^{(-)} f(x_i,y_i)$,或者 $\partial_x^{(+)} f_i$,$\partial_x^{(-)} f_i$,$\partial_y^{(+1)} f(x_i,y_i)$,$\partial_y^{(-1)} f(x_i,y_i)$ 统一写成 $\partial_y^{(+)} f(x_i,y_i)$,$\partial_y^{(-)} f(x_i,y_i)$,或者 $\partial_y^{(+)} f_i$,$\partial_y^{(-)} f_i$.

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2 含余项的常微分方程数值计算方法

设存在常微分方程初值问题

$$\begin{cases}\dfrac{dy}{dx}=f(x,y)\\ y(x_0)=y_0\end{cases} \tag{2.1}$$

$f(x,y)$ 为已知函数,对于 $f(x,y)$,通常假设,$f(x,y)$ 在某个区域内连续,且关于 y 三阶连续可微. 设 h 为步长. 现在构建带余项的数值计算方法.

包含余项的梯形公式的形式如下:

$$y_{i+1}=y_i+\frac{h}{2}[f(x_i,y_i)+f(x_{i+1},y_{i+1})]-\frac{h^3}{12}y'''(\xi)\ \xi\in(x_i,x_{i+1}) \tag{2.2}$$

(2.2)中最后一项称为梯形公式的余项,也就是截断误差. 关于(2.2)的推导可参阅相关的计算方法的书籍,例如本文的参考文献[1].

设 $y'''(\xi)=\dfrac{y''_{i+1}-y''_i}{h}$,则(2.2)变成:

$$y_{i+1}=y_i+\frac{h}{2}[f(x_i,y_i)+f(x_{i+1},y_{i+1})]-\frac{h^2}{12}(y''_{i+1}-y'_i) \tag{2.3}$$

y''_{i+1},y''_i 可以有不同的表达式,这样 $y''_{i+1}-y''_i$ 也可以有不同的表达式,现列举如下:

$$y''_{i+1}-y''_i=\frac{y'_{i+1}-y'_i}{h}-\frac{y'_i-y'_{i-1}}{h}$$

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$$= \frac{1}{h}(y'_{i+1} - 2y'_i + y'_{i-1})$$

$$= \frac{y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2}}{h^2} \quad (2.4)$$

$$y''_{i+1} - y''_i = \frac{y'_{i+2} - y'_{i+1}}{h} - \frac{y'_{i+1} - y'_i}{h}$$

$$= \frac{1}{h}(y'_{i+2} - 2y'_{i+1} + y'_i)$$

$$= \frac{1}{h}(f_{i+2} - 2f_{i+1} + f_i) \quad (2.5)$$

$$y''_{i+1} - y''_i = \frac{y'_{i+1} - y'_i}{h} - \frac{y'_i - y'_{i-1}}{h}$$

$$= \frac{1}{h}(y'_{i+1} - 2y'_i + y'_{i-1})$$

$$= \frac{1}{h}(f_{i+1} - 2f_i + f_{i-1}) \quad (2.6)$$

$$y''_{i+1} - y''_i = \frac{y'_{i+1} - y'_{i+\frac{1}{2}}}{\frac{1}{2}h} - \frac{y'_{i+\frac{1}{2}} - y'_i}{\frac{1}{2}h}$$

$$= \frac{2}{h}(y'_{i+1} - 2y'_{i+\frac{1}{2}} + y'_i)$$

$$= \frac{2}{h}(f_{i+1} - 2f_{i+\frac{1}{2}} + f_i) \quad (2.7)$$

另外由于

$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

所以 $y''_{i+1} - y''_i$ 也可以成以下形式.

$$y''_{i+1} - y''_i = \left(\frac{\partial f_{i+1}}{\partial x} + \frac{\partial f_{i+1}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\right) - \left(\frac{\partial f_i}{\partial x} + \frac{\partial f_i}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\right) \quad (2.8)$$

把以上公式带入到(2.3)中,并设 $\partial y = \mathrm{d}y$, $\partial x = \mathrm{d}x$ 得到以下公式.

$$y_{i+1} = y_i + \frac{h}{2}f(x_i, y_i) + \frac{h}{2}f(x_{i+1}, y_{i+1}) - \frac{1}{12}(y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2}) \quad (2.9)$$

$$y_{i+1} = y_i + \frac{h}{12}[-f(x_{i+2}, y_{i+2}) + 8f(x_{i+1}, y_{i+1}) + 5f(x_i, y_i)] \quad (2.10)$$

$$y_{i+1} = y_i + \frac{h}{12}[5f(x_{i+1}, y_{i+1}) + 8f(x_i, y_i) - f(x_{i-1}, y_{i-1})] \quad (2.11)$$

$$y_{i+1} = y_i + \frac{h}{3}f(x_i, y_i) + \frac{h}{3}f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}) + \frac{h}{3}f(x_{i+1}, y_{i+1}) \quad (2.12)$$

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1}] - \frac{h^2}{12\mathrm{d}x}[\partial_x f_{i+1} + \partial_y f_{i+1} - (\partial_x f_i + \partial_y f_i)] \quad (2.13)$$

整理(2.9)并设(2.13)中的 $dx = rh$,这样(2.9),(2.13)变成:

$$y_{i+1} = \frac{15}{13}y_i - \frac{3}{13}y_{i-1} + \frac{1}{13}y_{i-2} + \frac{6}{13}hf_i + \frac{6}{13}hf_{i+1} \quad (2.14)$$

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] - \frac{h}{12r}(\partial_x^{(r)} f_{i+1} + \partial_y^{(r)} f_{i+1} - \partial_x^{(r)} f_i - \partial_y^{(r)} f_i) \quad (2.15)$$

以上公式中,(2.11)就是两步 Adams 隐式方法. 而(2.10)中包含了 y_{i+2},不能直接计算,为此对(2.10)改进得到以下公式.

$$y_{i+1} = y_i + \frac{h}{12}[-f(x_{i+2}, y_{i+1} + hf_{i+1}) + 8f(x_{i+1}, y_{i+1}) + 5f(x_i, y_i)] \quad (2.16)$$

公式的构建中采用 y_{i+2} 的近似值,这也是新方法的特点之一. 对于(2.15)根据 r 的不同有无限多种计算公式,以下给出几个计算公式.

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{12}(\partial_x^{(+)} f_{i+1} + \partial_y^{(+)} f_{i+1} - \partial_x^{(+)} f_i - \partial_y^{(+)} f_i) \quad (2.17)$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{12}(\partial_x^{(-)} f_{i+1} + \partial_y^{(-)} f_{i+1} - \partial_x^{(+)} f_i - \partial_y^{(+)} f_i) \quad (2.18)$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}(\partial_x^{(+\frac{1}{2})} f_{i+1} + \partial_y^{(+\frac{1}{2})} f_{i+1} - \partial_x^{(+\frac{1}{2})} f_i - \partial_y^{(+\frac{1}{2})} f_i) \quad (2.19)$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}(\partial_x^{(-\frac{1}{2})} f_{i+1} + \partial_y^{(-\frac{1}{2})} f_{i+1} - \partial_x^{(+\frac{1}{2})} f_i - \partial_y^{(+\frac{1}{2})} f_i) \quad (2.20)$$

对于(2.17)可以有以下的具体计算公式.

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{12}[\partial_x^{(+)} f_{i+1} + f(x_{i+1}, y_{i+1} + hf_{i+1}) - f_{i+1} - \partial_x^{(+)} f_i - \partial_y^{(+)} f_i] \quad (2.21)$$

上面的公式中,有些需要计算 $f_{i+\frac{1}{2}}$,下面给出几种计算公式.

$$f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}) = f\left(x_{i+\frac{1}{2}}, y_i + \frac{h}{2}f_i\right) \quad (2.22)$$

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$$f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}) = f\left(x_{i+\frac{1}{2}}, y_{i+1} - \frac{h}{2}f_{i+1}\right) \tag{2.23}$$

$$f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}) = f\left(x_{i+\frac{1}{2}}, \frac{1}{2}(y_i + y_{i+1})\right) \tag{2.24}$$

$$f_{i+\frac{1}{2}} = f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}) = f\left(x_{i+\frac{1}{2}}, \frac{1}{2}(y_i + y_{i+1}) + \frac{h}{4}(f_i - f_{i+1})\right) \tag{2.25}$$

以上公式中(2.22),(2.23)采用了正向和反向 Euler 公式,(2.24)采用了插值公式,(2.25)采用了正向和反向 Euler 公式的算术平均. 把(2.22)~(2.25)代入(2.12)得到:

$$y_{i+1} = y_i + \frac{h}{3}f(x_i, y_i) + \frac{h}{3}f\left(x_{i+\frac{1}{2}}, y_i + \frac{h}{2}f_i\right) + \frac{h}{3}f(x_{i+1}, y_{i+1}) \tag{2.26}$$

$$y_{i+1} = y_i + \frac{h}{3}f(x_i, y_i) + \frac{h}{3}f\left(x_{i+\frac{1}{2}}, y_{i+1} - \frac{h}{2}f_{i+1}\right) + \frac{h}{3}f(x_{i+1}, y_{i+1}) \tag{2.27}$$

$$y_{i+1} = y_i + \frac{h}{3}f(x_i, y_i) + \frac{h}{3}f\left(x_{i+\frac{1}{2}}, \frac{1}{2}(y_i + y_{i+1})\right) + \frac{h}{3}f(x_{i+1}, y_{i+1}) \tag{2.28}$$

$$y_{i+1} = y_i + \frac{h}{3}f(x_i, y_i) + \frac{h}{3}f\left(x_{i+\frac{1}{2}}, \frac{1}{2}(y_i + y_{i+1})\right) + \frac{h}{4}(f_i - f_{i+1}) + \frac{h}{3}f(x_{i+1}, y_{i+1}) \tag{2.29}$$

把(2.26)和(2.27)做算术平均得到:

$$y_{i+1} = y_i + \frac{h}{3}f(x_i, y_i) + \frac{h}{6}\left[f\left(x_{i+\frac{1}{2}}, y_i + \frac{h}{2}f_i\right) + f\left(x_{i+\frac{1}{2}}, y_{i+1} - \frac{h}{2}f_{i+1}\right)\right] + \frac{h}{3}f(x_{i+1}, y_{i+1}) \tag{2.30}$$

把(2.22)~(2.25)代入(2.19),(2.20)得到:

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f\left(x_{i+1} + \frac{h}{2}, y_{i+1}\right) - f_{i+1} + f\left(x_{i+1}, y_{i+1} + \frac{h}{2}f_{i+1}\right) - f_{i+1} - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, y_i + \frac{h}{2}f_i\right) + f_i\right] \tag{2.31}$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f\left(x_{i+1} + \frac{h}{2}, y_{i+1}\right) - f_{i+1} + f\left(x_{i+1}, y_{i+1} + \frac{h}{2}f_{i+1}\right) - f_{i+1} - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, y_{i+1} - \frac{h}{2}f_{i+1}\right) + f_i\right] \tag{2.32}$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f\left(x_{i+1} + \frac{h}{2}, y_{i+1}\right) - f_{i+1} + f\left(x_{i+1}, y_{i+1} + \frac{h}{2}f_{i+1}\right) - f_{i+1} - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, \frac{1}{2}(y_i + y_{i+1})\right) + f_i\right] \tag{2.33}$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f\left(x_{i+1} + \frac{h}{2}, y_{i+1}\right) - f_{i+1} + f\left(x_{i+1}, y_{i+1} + \frac{h}{2}f_{i+1}\right) - f_{i+1} - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, \frac{1}{2}(y_i + y_{i+1}) + \frac{h}{4}(f_i - f_{i+1})\right) + f_i\right] \tag{2.34}$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f_{i+1} - f\left(x_{i+1} - \frac{h}{2}, y_{i+1}\right) + f_{i+1} - f(x_{i+1}, y_i + \frac{h}{2}f_i) - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, y_i + \frac{h}{2}f_i\right) + f_i\right] \tag{2.35}$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f_{i+1} - f\left(x_{i+1} - \frac{h}{2}, y_{i+1}\right) + f_{i+1} - f\left(x_{i+1}, y_{i+1} - \frac{h}{2}f_{i+1}\right) - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, y_{i+1} - \frac{h}{2}f_{i+1}\right) + f_i\right] \tag{2.36}$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f_{i+1} - f\left(x_{i+1} - \frac{h}{2}, y_{i+1}\right) + f_{i+1} - f\left(x_{i+1}, \frac{1}{2}(y_i + y_{i+1})\right) - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, \frac{1}{2}(y_i + y_{i+1})\right) + f_i\right] \tag{2.37}$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{6}\left[f_{i+1} - f\left(x_{i+1} - \frac{h}{2}, y_{i+1}\right) + f_{i+1} - f\left(x_{i+1}, \frac{1}{2}(y_i + y_{i+1}) + \frac{h}{4}(f_i - f_{i+1})\right) - f\left(x_i + \frac{h}{2}, y_i\right) + f_i - f\left(x_i, \frac{1}{2}(y_i + y_{i+1}) + \frac{h}{4}(f_i - f_{i+1})\right) + f_i\right] \tag{2.38}$$

(2.31)和(2.32),(2.35)和(2.36)做算术平

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均得到:

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{12}\Big[f\Big(x_{i+1} + \frac{h}{2}, y_{i+1}\Big) - f_{i+1} + f\Big(x_{i+1}, y_{i+1} + \frac{h}{2}f_{i+1}\Big) - 2f_{i+1} - f\Big(x_i + \frac{h}{2}, y_i\Big) + f_i - f\Big(x_i, y_i + \frac{h}{2}f_i\Big) + f_i - f\Big(x_i, y_{i+1} - \frac{h}{2}f_{i+1}\Big) + f_i\Big] \quad (2.39)$$

$$y_{i+1} = y_i + \frac{h}{2}(f_i + f_{i+1}) - \frac{h}{12}\Big[2\Big(f_{i+1} - f\Big(x_{i+1} - \frac{h}{2}, y_{i+1}\Big) + f_{i+1} - f\Big(x_i + \frac{h}{2}, y_i\Big) + f_i + f_i\Big) - f\Big(x_{i+1}, y_i + \frac{h}{2}f_i\Big) - f\Big(x_i, y_i + \frac{h}{2}f_i\Big) - f\Big(x_{i+1}, y_{i+1} - \frac{h}{2}f_{i+1}\Big) - f\Big(x_i, y_{i+1} - \frac{h}{2}f_{i+1}\Big)\Big] \quad (2.40)$$

3 稳定性分析

以下分析一些新方法的稳定性. 设存在以下实验方程

$$y' = ay$$

对于以上方程,如果 a 是实数,则 $a < 0$,如果 a 是复数,则 $\mathrm{Re}(a) < 0$. 设 $ah = \bar{a}$. 把实验方程代入(2.16),(2.18),(2.21)得到.

$$y_{i+1} = y_i - \frac{h}{12}\Big[a(y_{i+1} + ahy_{i+1})\Big] + \frac{8}{12}ahy_{i+1} + \frac{5}{12}ahy_i = \Big(1 + \frac{5}{12}\bar{a}\Big)y_i + \Big(\frac{7}{12}\bar{a} - \frac{1}{12}\bar{a}^2\Big)y_{i+1} \quad (3.1)$$

$$y_{i+1} = y_i + \frac{h}{2}(ay_i + ay_{i+1}) + \frac{h}{12}\Big[ay_{i+1} - ay_i - (ay_{i+1} - ay_i)\Big] = \Big(1 + \frac{\bar{a}}{2}\Big)y_i + \frac{\bar{a}}{2}y_{i+1} \quad (3.2)$$

$$y_{i+1} = y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{12}\Big[a(y_{i+1} + ahy_{i+1}) - ay_{i+1} - (ay_{i+1} - ay_i)\Big] = \Big(1 + \frac{5}{12}\bar{a}\Big)y_i + \Big(\frac{7}{12}\bar{a} - \frac{1}{12}a^2\Big)y_{i+1} \quad (3.3)$$

把实验方程代入(2.26),(2.27),(2.28),(2.29),(2.30)得到.

$$y_{i+1} = y_i + \frac{h}{3}ay_i + \frac{h}{3}ay_{i+1} + \frac{h}{3}a\Big(y_i + \frac{h}{2}ay_i\Big) = \Big(1 + \frac{2}{3}\bar{a} + \frac{\bar{a}^2}{6}\Big)y_i + \frac{\bar{a}}{3}y_{i+1} \quad (3.4)$$

$$y_{i+1} = y_i + \frac{h}{3}ay_i + \frac{h}{3}ay_{i+1} + \frac{h}{3}a\Big(y_{i+1} - \frac{h}{2}ay_{i+1}\Big) = \Big(1 + \frac{\bar{a}}{3}\Big)y_i + \Big(\frac{2}{3}\bar{a} - \frac{\bar{a}^2}{6}\Big)y_{i+1} \quad (3.5)$$

$$y_{i+1} = y_i + \frac{h}{3}ay_i + \frac{h}{3}ay_{i+1} + \frac{h}{3}a\Big(\frac{y_i + y_{i+1}}{2}\Big) = \Big(1 + \frac{\bar{a}}{2}\Big)y_i + \frac{\bar{a}}{2}y_{i+1} \quad (3.6)$$

$$y_{i+1} = y_i + \frac{h}{3}ay_i + \frac{h}{3}ay_{i+1} + \frac{h}{3}a\Big[\Big(\frac{1}{2}y_i + y_{i+1}\Big) + \frac{h}{4}(ay_i - ay_{i+1})\Big] = \Big(1 + \frac{\bar{a}}{2} + \frac{1}{12}\bar{a}^2\Big)y_i + \Big(\frac{\bar{a}}{2} - \frac{\bar{a}^2}{12}\Big)y_{i+1} \quad (3.7)$$

$$y_{i+1} = y_i + \frac{h}{3}ay_i + \frac{h}{3}ay_{i+1} + \frac{h}{6}\Big[a\Big(y_i + \frac{h}{2}ay_i\Big) + a\Big(y_{i+1} - \frac{h}{2}ay_{i+1}\Big)\Big] = y_i + \frac{1}{3}\bar{a}y_{i+1} + \frac{1}{3}\bar{a}^2y_i + \frac{1}{6}\bar{a}y_i + \frac{1}{12}\bar{a}^2y_i + \frac{1}{6}\bar{a}y_{i+1} - \frac{1}{12}\bar{a}^2y_{i+1} = \Big(1 + \frac{1}{2}\bar{a} + \frac{1}{12}\bar{a}^2\Big)y_i + \Big(\frac{\bar{a}}{2} - \frac{1}{12}\bar{a}^2\Big)y_{i+1} \quad (3.8)$$

(3.1) ~ (3.8) 移项整理后得到:

$$y_{i+1} = \frac{1 + \frac{5}{12}\bar{a}}{1 - \frac{7}{12}\bar{a} + \frac{1}{12}\bar{a}^2}y_i \quad (3.9)$$

$$y_{i+1} = \frac{1 + \frac{1}{2}\bar{a}}{1 - \frac{1}{2}\bar{a}}y_i \quad (3.10)$$

$$y_{i+1} = \frac{1 + \frac{5}{12}\bar{a}}{1 - \frac{7}{12}\bar{a} + \frac{1}{12}\bar{a}^2}y_i \quad (3.11)$$

$$y_{i+1} = \frac{1 + \frac{2}{3}\bar{a} + \frac{\bar{a}^2}{6}}{1 - \frac{\bar{a}}{3}}y_i \quad (3.12)$$

$$y_{i+1} = \frac{1 + \frac{\bar{a}}{3}}{1 - \frac{2}{3}\bar{a} + \frac{\bar{a}^2}{6}}y_i \quad (3.13)$$

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$$y_{i+1} = \frac{1+\frac{1}{2}\bar{a}}{1-\frac{1}{2}\bar{a}}y_i \tag{3.14}$$

$$y_{i+1} = \frac{1+\frac{1}{2}\bar{a}+\frac{1}{12}\bar{a}^2}{1-\frac{1}{2}\bar{a}+\frac{1}{12}\bar{a}^2}y_i \tag{3.15}$$

$$y_{i+1} = \frac{1+\frac{1}{2}\bar{a}+\frac{1}{12}\bar{a}^2}{1-\frac{1}{2}\bar{a}+\frac{1}{12}\bar{a}^2}y_i \tag{3.16}$$

(3.9) 和(3.11) 形式相同. 如果 a 是实数,则(3.9) ,(3.10) ,(3.11) ,(3.13) ,(3.14) ,(3.15) ,(3.16) 是无条件稳定的,如果 a 是复数,(3.9) ,(3.10) ,(3.11) ,(3.13) ,(3.14) ,(3.15) ,(3.16) 是 A 稳定的,其中(3.9) ,(3.11) ,(3.13) 是 L 稳定的. 对于(3.12) 稳定的条件为:

$$-1 < \frac{1+\frac{2\bar{a}}{3}+\frac{\bar{a}^2}{6}}{1-\frac{\bar{a}}{3}} < 1$$

求解上式. 左边的不等式无条件成立. 对于右边的不等式有:

$$1+\frac{2\bar{a}}{3}+\frac{\bar{a}^2}{6} < 1-\frac{\bar{a}}{3}$$

上式成立的稳定的条件为: $\bar{a} > -6$. 这样得到(3.12) 稳定的条件为: $\bar{a} > -6$.

把实验方程代入(2.31) ~(2.38) 得到.

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left[a\left(y_{i+1} + \frac{h}{2}ay_{i+1}\right) - ay_{i+1} - a\left(y_i + \frac{h}{2}ay_i\right) + ay_i\right] \\ &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left(\frac{h}{2}a^2y_{i+1} - \frac{h}{2}a^2y_i\right) \\ &= \left(1+\frac{1}{2}\bar{a}+\frac{1}{12}\bar{a}^2\right)y_i + \frac{1}{2}\bar{a}y_{i+1} - \frac{1}{12}\bar{a}^2y_{i+1} \end{aligned} \tag{3.17}$$

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left[a\left(y_{i+1} + \frac{h}{2}ay_{i+1}\right) - ay_{i+1} - a\left(y_{i+1} - \frac{h}{2}ay_{i+1}\right) + ay_i\right] \\ &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left(\frac{h}{2}a^2y_{i+1} - ay_{i+1} + \frac{h}{2}a^2y_{i+1} + ay_i\right) \\ &= \left(1+\frac{1}{3}\bar{a}\right)y_i + \frac{2}{3}\bar{a}y_{i+1} - \frac{1}{6}\bar{a}^2y_{i+1} \end{aligned} \tag{3.18}$$

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left[a\left(y_{i+1} + \frac{h}{2}ay_{i+1}\right) - ay_{i+1} - a\frac{1}{2}(y_i + y_{i+1}) + ay_i\right] \\ &= \left(1+\frac{5}{12}\bar{a}\right)y_i + \frac{7}{12}\bar{a}y_{i+1} - \frac{1}{12}\bar{a}^2y_{i+1} \end{aligned} \tag{3.19}$$

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left[a\left(y_{i+1} + \frac{h}{2}ay_{i+1}\right) - ay_{i+1} - a\left(\frac{1}{2}(y_i + y_{i+1}) + \frac{h}{4}(ay_i - ay_{i+1})\right) + ay_i\right] \\ &= \left(1+\frac{5}{12}\bar{a}+\frac{1}{24}\bar{a}^2\right)y_i + \frac{7}{12}\bar{a}y_{i+1} - \frac{3}{24}\bar{a}^2y_{i+1} \end{aligned} \tag{3.20}$$

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left[ay_{i+1} - a\left(y_i + \frac{h}{2}ay_i\right) - a\left(y_i + \frac{h}{2}ay_i\right) + ay_i\right] \\ &= y_i + \frac{\bar{a}}{2}y_i + \frac{\bar{a}}{2}y_{i+1} - \frac{\bar{a}}{6}y_{i+1} + \frac{\bar{a}^2}{6}y_i + \frac{\bar{a}}{6}y_i \\ &= \left(1+\frac{2\bar{a}}{3}+\frac{\bar{a}^2}{6}\right)y_i + \frac{\bar{a}}{3}y_{i+1} \end{aligned} \tag{3.21}$$

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(ay_i + ay_{i+1}) - \frac{h}{6}\left[ay_{i+1} - a\left(y_{i+1} - \frac{h}{2}ay_{i+1}\right) - a\left(y_{i+1} - \frac{h}{2}ay_{i+1}\right) + ay_i\right] \\ &= y_i + \frac{\bar{a}}{2}y_i + \frac{\bar{a}}{2}y_{i+1} - \frac{\bar{a}^2}{12}y_{i+1} + \frac{\bar{a}}{6}y_{i+1} - \frac{\bar{a}^2}{12}y_{i+1} - \frac{\bar{a}}{6}y_i \\ &= \left(1+\frac{\bar{a}}{3}\right)y_i + \left(\frac{2\bar{a}}{3} - \frac{\bar{a}^2}{6}\right)y_{i+1} \end{aligned} \tag{3.22}$$

$$\begin{aligned} y_{i+1} &= y_i + \frac{\bar{a}}{2}y_i + \frac{\bar{a}}{2}y_{i+1} - \frac{h}{6}\left[ay_{i+1} - \frac{a}{2}(y_i + y_{i+1}) - \frac{a}{2}(y_i + y_{i+1}) + ay_i\right] \\ &= y_i + \frac{\bar{a}}{2}y_i + \frac{\bar{a}}{2}y_{i+1} \end{aligned} \tag{3.23}$$

$$\begin{aligned} y_{i+1} &= y_i + \frac{\bar{a}}{2}y_i + \frac{\bar{a}}{2}y_{i+1} - \frac{h}{6}\left[ay_{i+1} - a\left(\frac{1}{2}(y_i + y_{i+1}) + \frac{h}{4}(ay_i - ay_{i+1})\right) - a\left(\frac{1}{2}(y_i + y_{i+1}) + \frac{h}{4}(ay_i - ay_{i+1})\right) + ay_i\right] \\ &= y_i + \frac{\bar{a}}{2}y_i + \frac{\bar{a}}{2}y_{i+1} - \frac{h}{6}\left[ay_{i+1} - ay_i - ay_{i+1} + ay_i - \frac{ah}{2}(ay_i - ay_{i+1})\right] \\ &= \left(1+\frac{\bar{a}}{2}+\frac{1}{12}\bar{a}^2\right)y_i + \left(\frac{\bar{a}}{2} - \frac{1}{12}\bar{a}^2\right)y_{i+1} \end{aligned} \tag{3.24}$$

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(3.17) ~(3.24) 整理后得到:

$$y_{i+1} = \frac{1 + \frac{1}{2}\bar{a} + \frac{1}{12}\bar{a}^2}{1 - \frac{1}{2}\bar{a} + \frac{1}{12}\bar{a}^2} y_i \tag{3.25}$$

$$y_{i+1} = \frac{1 + \frac{1}{3}\bar{a}}{1 - \frac{2}{3}\bar{a} + \frac{1}{6}\bar{a}^2} y_i \tag{3.26}$$

$$y_{i+1} = \frac{1 + \frac{5}{12}\bar{a}}{1 - \frac{7}{12}\bar{a} + \frac{1}{12}\bar{a}^2} y_i \tag{3.27}$$

$$y_{i+1} = \frac{1 + \frac{5}{12}\bar{a} + \frac{1}{24}\bar{a}^2}{1 - \frac{7}{12}\bar{a} + \frac{3}{24}\bar{a}^2} y_i \tag{3.28}$$

$$y_{i+1} = \frac{1 + \frac{2}{3}\bar{a} + \frac{1}{6}\bar{a}^2}{1 - \frac{1}{3}\bar{a}} y_i \tag{3.29}$$

$$y_{i+1} = \frac{1 + \frac{1}{3}\bar{a}}{1 - \frac{2}{3}\bar{a} + \frac{1}{6}\bar{a}^2} y_i \tag{3.30}$$

$$y_{i+1} = \frac{1 + \frac{1}{2}\bar{a}}{1 - \frac{1}{2}\bar{a}} \tag{3.31}$$

$$y_{i+1} = \frac{1 + \frac{1}{2}\bar{a} + \frac{1}{12}\bar{a}^2}{1 - \frac{1}{2}\bar{a} + \frac{1}{12}\bar{a}^2} y_i \tag{3.32}$$

以上各式中除(3.29)外,都是 A 稳定的,其中(3.26),(3.27)和(3.30)还是 L 稳定的. 当 a 是负实数时,(3.29)的稳定条件为 $\bar{a} > -6$.

4 数据计算

以下通过实际计算来观察一些新方法的精度. 为了便于对照给出了一些传统计算方法的计算结果. 传统方法的计算过程可参阅参考文献[2].

例 1: 求解下列常微分方程初值问题

$$y' = x - y + 1, y(0) = 1, h = 0.1, 0 \leqslant x \leqslant 1.$$

本例有精确解 $y = x + e^{-x}$.

以下采用多种方法进行求解,以便比较各种方法的精确度. 其中 Runge-Kutta 方法采用经典四级四阶 Runge-Kuta 方法,Adams 隐式公式为四阶 Adams 隐式公式,即为:

$$y_{i+1} = y_i + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2})$$

计算结果见下面的表格. 本例形式简单,有些公式具有相同的计算形式和计算结果.

表 1 传统公式计算结果及精确解

x	精确解	显式 Euler 公式	隐式 Euler 公式	梯形公式	RK 方法	Adams 公式
0.1	1.004 837 418	1.000 000 000	1.009 090 909	1.004 827 418	1.004 837 500	
0.2	1.018 730 753	1.010 000 000	1.026 446 289	1.018 730 753	1.018 730 901	
0.3	1.040 818 221	1.029 000 000	1.051 314 801	1.040 936 613	1.040 818 422	1.040 818 006
0.4	1.070 320 046	1.056 100 000	1.083 013 455	1.070 096 589	1.070 320 288	1.070 319 661
0.5	1.106 530 660	1.090 490 000	1.120 921 323	1.106 277 930	1.106 530 934	1.106 530 138
0.6	1.148 811 636	1.131 441 000	1.164 473 970	1.148 537 233	1.148 811 934	1.148 811 007
0.7	1.196 585 304	1.178 296 900	1.213 158 118	1.196 295 644	1.196 585 618	1.196 584 593
0.8	1.249 328 964	1.230 467 610	1.266 507 380	1.249 029 439	1.249 339 289	1.249 328 193
0.9	1.306 656 966	1.287 420 489	1.324 097 618	1.306 264 773	1.306 569 991	1.306 568 846
1.0	1.367 879 441	1.348 678 440	1.385 543 289	1.367 572 929	1.367 879 774	1.367 878 599

表 2 部分新公式的计算结果(一)

x	(2.14)	(2.26)	(2.27)	(2.30)	(2.28)	(2.29)
0.1	1.004 837 418	1.004 838 709	1.004 836 193	1.004 837 451	1.004 761 904	1.004 837 430
0.2	1.018 730 753	1.018 733 090	1.018 728 536	1.018 730 813	1.018 594 104	1.018 730 775
0.3	1.040 824 308	1.040 821 393	1.040 815 212	1.040 818 303	1.040 632 761	1.040 818 251
0.4	1.070 331 999	1.070 323 873	1.070 316 417	1.070 320 145	1.070 096 307	1.070 320 083
0.5	1.106 546 957	1.106 534 988	1.106 526 555	1.106 530 772	1.106 277 611	1.106 530 701
0.6	1.148 831 213	1.148 816 336	1.148 807 179	1.148 811 758	1.148 536 886	1.148 811 681
0.7	1.196 607 396	1.196 590 265	1.196 580 599	1.196 585 432	1.196 295 278	1.196 585 352
0.8	1.249 352 928	1.249 334 095	1.249 324 099	1.249 329 097	1.249 029 061	1.249 329 014
0.9	1.306 594 940	1.306 574 883	1.306 564 707	1.306 569 795	1.306 264 388	1.306 569 710
1.0	1.367 905 570	1.367 884 692	1.367 874 462	1.367 879 577	1.367 572 542	1.367 879 492

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表 3　部分新公式的计算结果(二)

x	(2.16)	(2.21)	(2.31)	(2.32)	(2.39)	(2.40)
0.1	1.004 799 370	1.004 799 370	1.004 837 430	1.004 836 193	1.004 836 812	1.004 837 451
0.2	1.018 661 900	1.018 661 900	1.018 730 775	1.018 728 536	1.018 729 656	1.018 730 813
0.3	1.040 724 772	1.040 724 772	1.040 818 251	1.040 815 212	1.040 816 732	1.040 818 303
0.4	1.070 207 308	1.070 207 308	1.070 320 083	1.070 316 417	1.070 318 250	1.070 320 145
0.5	1.106 403 150	1.106 403 150	1.106 530 701	1.106 526 555	1.106 528 628	1.106 530 772
0.6	1.148 673 188	1.148 673 188	1.148 811 681	1.148 807 179	1.148 809 430	1.148 811 758
0.7	1.196 439 156	1.196 439 156	1.196 585 352	1.196 580 599	1.196 582 975	1.196 585 432
0.8	1.249 177 835	1.249 177 835	1.249 329 014	1.249 324 099	1.249 326 556	1.249 329 097
0.9	1.306 415 823	1.306 415 823	1.306 569 710	1.306 564 707	1.306 567 209	1.306 569 795
1.0	1.367 724 781	1.367 724 781	1.367 879 492	1.367 874 462	1.367 876 977	1.367 879 577

从计算结果可以看出: 以上公式的计算结果除了(2.28)外,其他都明显好于梯形公式,其中公式(2.29),(2.30),(2.31),(2.40)的精度好于四级四阶 Runge-Kutta 方法和四阶 Adams 隐式方法.

5　结　论

本文给出了一种包含余项的常微分方程数值计算方法,并给出了一些具体的计算公式. 从文中可以看出,这类方法稳定性好,精算精度高,是一类具有使用价值的新的常微分方程计算方法.

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A New Kind of Differential Equation Calculational Method with Remainder Terms

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Abstract

This paper raises a new thinking for ordinary differential equation calculational method, sets up some formulas, analyzes the characteristics, and gives some examples. This new kind of method which has higher stability and accuracy is a practical calculational method for ordinary differential equation.

Key words: calculational method for ordinary differential equation, ordinary differential equation, the remainder term.

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