DOI:10.19789/j.1004-9398.2015.01.002

线性规划的一个新的 Mehrotra 型预估 - 矫正算法*

刘长河 吴 丹
(河南科技大学 数学与统计学院,洛阳 471023)

摘 要

在线性规划的内点算法中,理论和实践之间存在着数值效果好的算法具有较差复杂性的矛盾. 目前大多数内点算法软件的执行采用 Mehrotra 型预估 - 矫正算法. 本文提出了求解线性规划问题的一个新的 Mehrotra 型预估 - 矫正算法,证明了该算法的迭代复杂性是 $O(\sqrt{n}L)$,这是内点算法所具有的最好的复杂性结果.

关键词: 线性规划, 内点法, Mehrotra 型预估 - 矫正算法, 多项式复杂性.
中图分类号: O221.1

0 引 言

线性规划问题和它的对偶问题定义为:

$$(\mathrm{P})\quad \min c^T x,\quad s.t.\ Ax = b, x \geqslant 0,$$
$$(\mathrm{D})\quad \max b^T y,\quad s.t.\ A^T y + s = c, s \geqslant 0, \qquad (1)$$

其中 $A \in R^{m\times n}, c, x, s \in R^n, b, y \in R^m$. 记线性规划的原 - 对偶严格可行集为:

$$F_{++} := \{(x, y, s) \in R^n \times R^m \times R^n : Ax = b, A^T y + s = c, (x, s) > 0\}.$$

求解线性规划等价于求解它的 KKT 最优性条件:

$$Ax = b, A^T y + s = c, x \geqslant 0, s \geqslant 0, xs = 0. \qquad (2)$$

扰动互补条件 $xs = 0$ 得

$$Ax = b, A^T y + s = c, x \geqslant 0, s \geqslant 0, xs = \mu e. \qquad (3)$$

当严格可行集 F_{++} 非空时,对任意 $\mu > 0$,扰动系统(3)存在唯一解,称之为线性规划的 μ - 中心. 所有 μ - 中心的集合称为中心路径,记作 $C := \{(x, y, s) \in F_{++} : xs = \mu e, \mu > 0\}$. 当 $\mu \to 0$ 时,中心路径的极限 (x^*, y^*, s^*) 即是线性规划问题(P)和(D)的最优解.

路径跟踪算法的基本思想是用牛顿法求解方程组(3),逐渐减小 μ,并使得迭代点列包含于中心路径的某个邻域内,最终得到满足允许精度的最优解. 基于宽邻域 N_∞^- 的内点算法的复杂性是 $O(nL)$,基于窄邻域 N_2 的内点算法的复杂性为 $O(\sqrt{n}L)$. 但是,由于窄邻域极大地限制了迭代步长,使得该算法的实际计算效果不如宽邻域算法. Peng 等[1] 利用自 - 正则函数定义的宽邻域,提出了一个预估 - 矫正算法,证明其复杂性是 $O(\sqrt{n}\log nL)$. Ai 和 Zhang[2] 提出了求解单调线性互补问题的路径跟踪算法,他们利用新的宽邻域,并把牛顿方向分解成两个方向. 运用这些策略使得新的宽邻域算法具有 $O(\sqrt{n}L)$ 复杂性,其数值实验显示出算法具有良好的计算效果. 另一方面,目前大多数内点算法软件的执行采用 Mehrotra 型预估 - 矫正算法[3]. 基于宽邻域 N_∞^-,文献[4]研究了线性规划的 Mehrotra 型预估 - 矫正算法的收敛性. 作者首先提出了一类 Mehrotra 型预估 - 矫正算法,证明了该算法的迭代复杂性为 $O(n^2L)$,然后,通过修改矫正策略,把复杂性降低到 $O(nL)$. 文献[5]针对[2]中的路径跟踪算法增加一个二阶矫正步,提出线性规划的宽邻域预估 - 矫正算法,其迭代复杂性为 $O(\sqrt{n}L)$. 本文受[4,5]的启发,采用类似的矫正策略,提出了一个新的 Mehrotra 型预估 - 矫正算法,并证明该算法的迭代复杂性是 $O(\sqrt{n}L)$.

本文中的记法和约定: e 表示分量全为1的列向量,$\|\cdot\|_p$ 表示向量的 L_p - 范数,$\|\cdot\|$ 表示向量的2 - 范数. 对于 $a \in R$,记 $a^+ := \max\{a, 0\}$, $a^- :=$

收稿日期: 2014-01-21
* 国家自然科学基金(61301229),河南科技大学自然科学基金(2013QN030),河南科技大学自然科学基金(2014QN039).

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$\min\{a,0\}$. 对于向量 $x,s \in R^n$, xs 表示分量乘积,类似的方式记 $1/(xs)$, $(xs)^{-0.5}$ 等. 记指标集 $I=\{1,2,\cdots,n\}$. 我们假设 $F_{++} \neq \varphi$ 且 $\mathrm{rank}(A)=m$.

1 Mehrotra 型预估-矫正算法

在线性规划的内点算法中,通常使用的窄邻域和宽邻域分别为:

$$N_2(\tau):=\{(x,y,s) \in F_{++}: \|xs-\mu e\| \leqslant \tau\mu\},$$

$$N_\infty^-(1-\tau):=\{(x,y,s) \in F_{++}: xs \geqslant \tau\mu\},$$

其中 $\tau \in (0,1)$ 是给定常数, $\mu = x^T s/n$. Ai 和 Zhang 在 [2] 中提出了一个新的宽邻域:

$$N(\tau_1,\tau_2):=\{(x,y,s) \in F_{++}: \|(\tau_1\mu e-xs)^+\| \leqslant (\tau_1-\tau_2)\mu\},\ 0<\tau_2<\tau_1<1.$$

容易证明邻域的包含关系:

$$N_\infty^-(1-\tau_1) \subseteq N(\tau_1,\tau_2) \subseteq N_\infty^-(1-\tau_2),\ \forall 0<\tau_2<\tau_1<1. \tag{4}$$

设当前迭代点为 (x,y,s),令 $0 \leqslant \tau \leqslant 1$. [2] 中的路径跟踪算法把牛顿方向分解为两个方向:

$$\begin{cases} A\Delta x_- = 0, \\ A^T\Delta y_- + \Delta s_- = 0, \\ s\Delta x_- + x\Delta s_- = (\tau\mu e - xs)^- \end{cases} \tag{5}$$

和

$$\begin{cases} A\Delta x_+ = 0, \\ A^T\Delta y_+ + \Delta s_+ = 0, \\ s\Delta x_+ + x\Delta s_+ = (\tau\mu e - xs)^+. \end{cases} \tag{6}$$

则算法产生的迭代点为:

$$(x(\alpha),y(\alpha),s(\alpha)):=(x,y,s)+\alpha_1(\Delta x_-,\Delta y_-,\Delta s_-)+\alpha_2(\Delta x_+,\Delta y_+,\Delta s_+),$$

其中,步长向量 $\alpha:=(\alpha_1,\alpha_2)$ 由平面搜索确定:

$$\min_{0\leqslant\alpha_1,\alpha_2\leqslant 1}\{x(\alpha)^T s(\alpha): (x(\alpha),y(\alpha),s(\alpha)) \in N(\tau_1,\tau_2)\}. \tag{7}$$

本文采用 [4] 中改进算法的方式计算矫正方向 $(\Delta x^c,\Delta y^c,\Delta s^c)$:

$$\begin{cases} A\Delta x^c = 0, \\ A^T\Delta y^c + \Delta s^c = 0, \\ s\Delta x^c + x\Delta s^c = -\alpha_a\Delta x_-\ \Delta s_-, \end{cases} \tag{8}$$

其中, $\alpha_a \in [0,1]$ 是沿方向 $(\Delta x_-,\Delta y_-,\Delta s_-)$ 的最大可行步长,即

$$\alpha_a:=\arg\max\{\alpha \in [0,1]: x+\alpha\Delta x_- \geqslant 0, s+\alpha\Delta s_- \geqslant 0\}. \tag{9}$$

则新的迭代点为:

$$\begin{aligned}(x(\alpha),y(\alpha),s(\alpha)) &:= (x,y,s)+(\Delta x(\alpha),\Delta y(\alpha),\Delta s(\alpha)) \\ &:= (x,y,s)+\alpha_1(\Delta x_-,\Delta y_-,\Delta s_-)+(\Delta x^c,\Delta y^c,\Delta s^c)) \\ &\quad +\alpha_2(\Delta x_+,\Delta y_+,\Delta s_+).\end{aligned} \tag{10}$$

算法 1 (Mehrotra 型预估-矫正算法)

步骤 0 输入 $\varepsilon>0$, $0<\tau_2<\tau_1<1$, $0 \leqslant \tau \leqslant 1$ 及 $(x^0,y^0,s^0) \in N(\tau_1,\tau_2)$, $\mu_0=(x^0)^T s^0/n$. 置 $k:=0$.

步骤 1 若 $(x^k)^T s^k \leqslant \varepsilon$ 则停.

步骤 2 (预估步) 由(5)和(6)解得方向 $(\Delta x_-^k,\Delta y_-^k,\Delta s_-^k)$ 和 $(\Delta x_+^k,\Delta y_+^k,\Delta s_+^k)$,由(9)计算 α_a^k.

步骤 3 (矫正步) 由(8)解得矫正方向 $(\Delta x^{c,k},\Delta y^{c,k},\Delta s^{c,k})$. 计算步长 $\alpha^k=(\alpha_1^k,\alpha_2^k)$ 使得对偶间隙充分减小,并且满足 $(x(\alpha^k),y(\alpha^k),s(\alpha^k)) \in N(\tau_1,\tau_2)$.

步骤 4 令 $(x^{k+1},y^{k+1},s^{k+1}):=(x(\alpha^k),y(\alpha^k),s(\alpha^k))$, $\mu_{k+1}:=(x^{k+1})^T s^{k+1}/n$. 置 $k:=k+1$,转步骤 1.

引理 1 设 $(x,y,s) \in F_{++}$,方向 $(\Delta x_-,\Delta y_-,\Delta s_-)$ 满足方程组(5),预估步长 α_a 满足(9). 记指标集 $I_+=\{i \in I:(\Delta x_-)_i(\Delta s_-)_i>0\}$, $I_-=I\backslash I_+$. 则

(i) $(\Delta x_-)_i(\Delta s_-)_i \leqslant \dfrac{x_i s_i}{4}, \forall i \in I_+$,

(ii) $-(\Delta x_-)_i(\Delta s_-)_i \leqslant \dfrac{x_i s_i}{\alpha_a^2}, \forall i \in I_-$.

证明: 对任意 $i \in I_+$,由(5)得 $s_i(\Delta x_-)_i + x_i(\Delta s_-)_i = (\tau\mu - x_i s_i)^- \leqslant 0$,故 $s_i(\Delta x_-)_i<0$ 且 $s_i(\Delta s_-)_i<0$. 由基本不等式得 $2\sqrt{x_i s_i(\Delta x_-)_i(\Delta s_-)_i} \leqslant -s_i(\Delta x_-)_i - x_i(\Delta s_-)_i = -(\tau\mu - x_i s_i)^- = (x_i s_i - \tau\mu)^+ \leqslant x_i s_i$. 由此可得 $(\Delta x_-)_i(\Delta s_-)_i \leqslant x_i s_i/4$.

由(9)和(5)可得 $(x_i+\alpha_a(\Delta x_-)_i)(s_i+\alpha_a(\Delta s_-)_i) = x_i s_i + \alpha_a(\tau\mu - x_i s_i)^- + \alpha_a^2(\Delta x_-)_i(\Delta s_-)_i \geqslant 0$. 因此

$$-(\Delta x_-)_i(\Delta s_-)_i \leqslant (x_i s_i + \alpha_a(\tau\mu - x_i s_i)^-)/\alpha_a^2 \leqslant x_i s_i/\alpha_a^2.$$

利用引理 1(i)与正交性 $\Delta x_-^T\Delta s_- = 0$ 可得下面结果.

引理 2 设 $(x,y,s) \in F_{++}$,方向 $(\Delta x_-,\Delta y_-,\Delta s_-)$ 满足方程组(5). 则

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$$\sum_{i\in I_-}|(\Delta x_-)_i(\Delta s_-)_i| = \sum_{i\in I_+}(\Delta x_-)_i(\Delta s_-)_i \leqslant \frac{x^Ts}{4}.$$

引理 3[2]　设 $p,q\in R^n$, $p^Tq\geqslant 0$, 令 $r=p+q$, 则 $\|(pq)^-\|_1 \leqslant \|(pq)^+\|_1 \leqslant 4^{-1}\|r\|^2$.

下面给出两个常用结论

$$-x^Ts \leqslant e^T(\tau\mu e - xs)^- \leqslant -(1-\tau)x^Ts \quad (11)$$

$$e^T(\tau\mu e - xs)^+ \leqslant \sqrt{n}\,\|(\tau\mu e - xs)^+\|. \quad (12)$$

为了简单起见, 记 $\beta := (\tau_1-\tau_2)/\tau_1$. 则有 $\beta\in(0,1)$, $\tau_1-\tau_2=\beta\tau_1$, $\tau_2=(1-\beta)\tau_1$. 引入新的邻域记号 $N(\tau_1,\beta)$ 代替 $N(\tau_1,\tau_2)$, 即 $N(\tau_1,\beta) := \{(x,y,s)\in F_{++}: \|(\tau_1\mu e - xs)^+\| \leqslant \beta\tau_1\mu\}$. 本文中令 $\tau=\tau_1$, 并记 $h(\alpha) := xs + \alpha_1(\tau_1\mu e - xs)^- + \alpha_2(\tau_1\mu e - xs)^+ - \alpha_1\alpha_a(\Delta x_-\Delta s_-)$.

引理 4　设 $(x,y,s)\in N(\tau_1,\beta)$, $\beta\leqslant 1/2$. 若 $\alpha_1=\alpha_2\sqrt{\frac{\beta\tau_1}{2n}}$, 则

$$\|(\Delta x(\alpha)\Delta s(\alpha))^-\|_1 \leqslant \|(\Delta x(\alpha)\Delta s(\alpha))^+\|_1 \leqslant \frac{2}{3}\alpha_2^2\beta\tau_1\mu.$$

证明: 记 $r=(xs)^{-0.5}[\alpha_1(\tau_1\mu e - xs)^- + \alpha_2(\tau_1\mu e - xs)^+] - \alpha_1\alpha_a(xs)^{-0.5}(\Delta x_-\Delta s_-)$, $p=(x^{-0.5}s^{0.5})\Delta x(\alpha)$, $q=(x^{0.5}s^{-0.5})\Delta s(\alpha)$. 则由 (5), (6) 和 (8) 可知 $p^Tq=\Delta x(\alpha)^T\Delta s(\alpha)=0$ 且 $p+q=r$. 由引理 3 得

$$\begin{aligned}\|(\Delta x(\alpha)\Delta s(\alpha))^-\|_1 &\leqslant \|(\Delta x(\alpha)\Delta s(\alpha))^+\|_1 \\ &\leqslant 4^{-1}\|(xs)^{-0.5}(\alpha_1(\tau_1\mu e - xs)^- + \alpha_2(\tau_1\mu e - xs)^+) - \alpha_1\alpha_a(xs)^{-0.5}(\Delta x_-\Delta s_-)\|^2 \\ &\leqslant 4^{-1}(\|(\alpha_1(\tau_1\mu e - xs)^- + \alpha_2(\tau_1\mu e - xs)^+)/\sqrt{xs}\| + \alpha_1\|(\alpha_a\Delta x_-\Delta s_-)/\sqrt{xs}\|)^2. \quad (13)\end{aligned}$$

对于 (13) 中前一个范数, 由正交性得

$$\begin{aligned}&\|(\alpha_1(\tau_1\mu e - xs)^- + \alpha_2(\tau_1\mu e - xs)^+)/\sqrt{xs}\|^2 \\ &= \alpha_1^2\|(\tau_1\mu e - xs)^-/\sqrt{xs}\|^2 + \alpha_2^2\|(\tau_1\mu e - xs)^+/\sqrt{xs}\|^2 \\ &\leqslant \alpha_1^2\|(\sqrt{xs} - \tau_1\mu/\sqrt{xs})^+\|^2 + \alpha_2^2\|(\tau_1\mu e - xs)^+\|^2/(\tau_2\mu) \\ &\leqslant \alpha_1^2\|\sqrt{xs}\|^2 + \alpha_2^2(^{\beta}\tau_1\mu)2/(\tau_2\mu) \\ &= \alpha_2^2\frac{\beta\tau_1}{2n}(n\mu) + \frac{\alpha_2^2\beta^2\tau_1\mu}{1-\beta} \\ &\leqslant \frac{3}{2}\alpha_2^2\beta\tau_1\mu, \quad (14)\end{aligned}$$

其中第一个不等号由包含关系 (4) 得到. 对于 (13) 中后一个范数, 利用引理 1 和引理 2 可得

$$\begin{aligned}\|(\alpha_a\Delta x_-\Delta s_-)/\sqrt{xs}\|_2 &= \sum_{i\in I_+}\frac{\alpha_a^2((\Delta x_-)_i(\Delta s_-)_i)^2}{x_is_i} + \sum_{i\in I_-}\frac{\alpha_a^2((\Delta x_-)_i(\Delta s_-)_i)^2}{x_is_i} \\ &\leqslant \sum_{i\in I_+}\frac{\alpha_a^2(x_is_i)^2}{16x_is_i} + \sum_{i\in I_-}\frac{x_is_i(-(\Delta x_-)_i(\Delta s_-)_i)}{x_is_i} \\ &= \frac{\alpha_a^2}{16}\sum_{i\in I_+}x_is_i + \sum_{i\in I_-}(-(\Delta x_-)_i(\Delta s_-)_i) \\ &\leqslant \frac{1}{16}n\mu + \frac{1}{4}n\mu = \frac{5}{16}n\mu. \quad (15)\end{aligned}$$

把 (14), (15) 代入 (13) 得

$$\begin{aligned}&\|(\Delta x(\alpha)\Delta s(\alpha))^+\|_1 \\ &\leqslant \frac{1}{4}\left(\sqrt{\frac{3}{2}\alpha_2^2\beta\tau_1\mu} + \alpha_1\sqrt{\frac{5}{16}n\mu}\right)^2 \\ &= \frac{1}{4}\left(\alpha_2\sqrt{\frac{3}{2}\beta\tau_1\mu} + \alpha_2\sqrt{\frac{5}{32}\beta\tau_1\mu}\right)^2 \\ &\leqslant \frac{2}{3}\alpha_2^2\beta\tau_1\mu.\end{aligned}$$

由引理 1(i), 当 $\tau_1\mu - x_is_i\leqslant 0$ 时,

$$\begin{aligned}h(\alpha)_i &= x_is_i + \alpha_1(\tau_1\mu - x_is_i) - \alpha_1\alpha_a(\Delta x_-)_i(\Delta s_-)_i \\ &\geqslant x_is_i + \alpha_1\tau_1\mu - \alpha_1x_is_i - \frac{1}{4}\alpha_1x_is_i \\ &\geqslant \left(1 - \frac{5}{4}\alpha_1\right)x_is_i + \alpha_1\tau_1\mu. \quad (16)\end{aligned}$$

当 $\tau_1\mu - x_is_i > 0$ 时,

$$\begin{aligned}h(\alpha)_i &= x_is_i + \alpha_2(\tau_1\mu - x_is_i) - \alpha_1\alpha_a(\Delta x_-)_i(\Delta s_-)_i \\ &\geqslant x_is_i + \alpha_2\tau_1\mu - \alpha_2x_is_i - \frac{1}{4}\alpha_1x_is_i \\ &\geqslant \left(1 - \alpha_2 - \frac{1}{4}\alpha_1\right)x_is_i + \alpha_2\tau_1\mu. \quad (17)\end{aligned}$$

设 $(x,y,s)\in N(\tau_1,\beta)$, 由 (11), (12) 可得

$$\begin{aligned}\mu(\alpha) &= (x+\Delta x(\alpha))T(s+\Delta s(\alpha))/n \\ &= x^Ts/n + \alpha_1e^T(\tau_1\mu e - xs)^-/n + \alpha_2e^T(\tau_1\mu e - xs)^+/n \\ &\leqslant \mu - \alpha_1(1-\tau_1)\mu + \alpha_2\|(\tau_1\mu e - xs)^+\|/\sqrt{n} \\ &\leqslant \mu - \alpha_1(1-\tau_1)\mu + \alpha_2\beta\tau_1\mu/\sqrt{n}. \quad (18)\end{aligned}$$

类似可得

$$\begin{aligned}\mu(\alpha) &\geqslant \mu + \alpha_1e^T(\tau_1\mu e - xs)^-/n \\ &\geqslant \mu - \alpha_1x^Ts/n = (1-\alpha_1)\mu. \quad (19)\end{aligned}$$

引理 5　设 $(x,y,s)\in N(\tau_1,\beta)$, $\tau_1\leqslant 1/4$, $\beta\leqslant 1/2$. 若 $\alpha_1=\alpha_2\sqrt{\frac{\beta\tau_1}{2n}}$, 则

$$\|(\tau_1\mu(\alpha)e - h(\alpha))^+\| \leqslant (1-\alpha_2)\beta\tau_1\mu.$$

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net

证明: 当 $\tau_1\mu - x_is_i \leqslant 0$ 时,由(16),(18)可得

$$\begin{aligned}\tau_1\mu(\alpha) - h(\alpha)_i &\leqslant \tau_1(\mu - \alpha_1(1-\tau_1)\mu + \alpha_2\beta\tau_1\mu/\sqrt{n}) - \left(1 - \frac{5}{4}\alpha_1\right)x_is_i - \alpha_1\tau_1\mu\\ &\leqslant \tau_1(\mu - \alpha_1(1-\tau_1)\mu + \alpha_2\beta\tau_1\mu/\sqrt{n}) - \left(1 - \frac{5}{4}\alpha_1\right)\tau_1\mu - \alpha_1\tau_1\mu\\ &= -\frac{3}{4}\alpha_1\tau_1\mu + \alpha_1\tau_1^2\mu + \alpha_2\beta\tau_1^2\mu/\sqrt{n}\\ &= \left(-\frac{3}{4} + \tau_1 + \sqrt{2\beta\tau_1}\right)\alpha_1\tau_1\mu \leqslant 0.\end{aligned}$$

当 $\tau_1\mu - x_is_i > 0$ 时,由(17),(18)及上面结果得

$$\begin{aligned}\tau_1\mu(\alpha) - h(_\alpha &\leqslant \tau_1(\mu - \alpha_1(1-\tau_1)\mu + \alpha_2\beta\tau_1\mu/\sqrt{n}) - \left(1 - \alpha_2 - \frac{1}{4}\alpha_1\right)x_is_i - \alpha_2\tau_1\mu\\ &= \left(1 - \alpha_2 - \frac{1}{4}\alpha_1\right)(\tau_1\mu - x_is_i) - \frac{3}{4}\alpha_1\tau_1\mu + \alpha_1\tau_1^2\mu + \alpha_2\beta\tau_1^2\mu/\sqrt{n}\\ &\leqslant \left(1 - \alpha_2 - \frac{1}{4}\alpha_1\right)(\tau_1\mu - x_is_i)\\ &\leqslant (1-\alpha_2)(\tau_1\mu - x_is_i).\end{aligned}$$

因此 $\| (\tau_1\mu(\alpha)e - h(\alpha))^+ \| \leqslant (1-\alpha_2)\| (\tau_1\mu e - xs)^+ \| \leqslant (1-\alpha_2)\beta\tau_1\mu.$

引理 6 设 $(x,y,s) \in N(\tau_1,\beta)$,$\tau_1 \leqslant 1/4$,$\beta \leqslant 1/2$. 若 $\alpha_1 = \alpha_2\sqrt{\frac{\beta\tau_1}{2n}}$,则

$$\| (\tau_1\mu(\alpha)e - x(\alpha)s(\alpha))^+ \| \leqslant \beta\tau_1\mu(\alpha).$$

证明: 由引理 4,引理 5 及(19)可得

$$\begin{aligned}&\| (\tau_1\mu(\alpha)e - x(\alpha)s(\alpha))^+ \|\\ =\ &\| (\tau_1\mu(\alpha)e - h(\alpha) - \Delta x(\alpha)\Delta s(\alpha))^+ \|\\ \leqslant\ &\| (\tau_1\mu(\alpha)e - h(\alpha))^+ \| + \| (-\Delta x(\alpha)\Delta s(\alpha))^+ \|\\ \leqslant\ &(1-\alpha_2)\beta\tau_1\mu + \frac{2}{3}\alpha_2^2\beta\tau_1\mu\\ =\ &(1-\alpha_1)\beta\tau_1\mu + (\alpha_1 - \alpha_2 + \frac{2}{3}\alpha_2^2)\beta\tau_1\mu\\ \leqslant\ &(1-\alpha_1)\beta\tau_1\mu\\ \leqslant\ &\beta\tau_1\mu(\alpha).\end{aligned}$$

定理 1 设 $(x,y,s) \in N(\tau_1,\beta)$,$\tau_1 \leqslant 1/4$,$\beta \leqslant 1/2$. 若 $\alpha_1 = \alpha_2\sqrt{\frac{\beta\tau_1}{2n}}$,则

$$(x(\alpha),y(\alpha),s(\alpha)) \in N(\tau_1,\beta).$$

证明: 首先由引理 6 得 $\| (\tau_1\mu(\alpha)e - x(\alpha)s(\alpha))^+ \| \leqslant \beta\tau_1\mu(\alpha)$,则利用邻域包含关系(4)可得 $x(\alpha)s(\alpha) \geqslant (1-\beta)\tau_1\mu(\alpha) > 0$. 由迭代式(10)得

$$\begin{aligned}(x(\alpha),s(\alpha)) &= (x,s) + (\Delta x(\alpha),\\ \Delta s(\alpha)) &= (x,s) + \alpha_1(\Delta x_- + \Delta x^c, \Delta s_- + \Delta s^c) + \alpha_2(\Delta x_+, \Delta s_+).\end{aligned}$$

设 $z = \alpha_1((\tau\mu e - xs)^- - \alpha_a\Delta x_-\Delta s_-) + \alpha_2(\tau\mu e - xs)^+$,则由引理 1(i)得

$$\begin{aligned}z + 2xs &= \alpha_1((\tau\mu e - xs)^- - \alpha_a\Delta x_-\Delta s_-) + \alpha_2(\tau\mu e - xs)^+ + 2xs\\ &\geqslant \alpha_1\left(-xs - \frac{\alpha_a}{4}xs\right) + 2xs \geqslant 0.\end{aligned}$$

因此,由[5, Lemma 3.4]可得 $(x(\alpha),s(\alpha)) = (x + \Delta x(\alpha), s + \Delta s(\alpha)) > 0$.

综上即得 $(x(\alpha),y(\alpha),s(\alpha)) \in N(\tau_1,\beta)$.

2 算法的多项式复杂性

定理 2 设 $(x^0,y^0,s^0) \in N(\tau_1,\beta)$,$\tau = \tau_1 \leqslant 1/4$,$\beta \leqslant 1/2$,利用(7)计算迭代步长,则算法 1 最多需要 $O\left(\sqrt{n}\log\frac{(x^0)^Ts^0}{\varepsilon}\right)$ 次迭代后终止.

证明: 由定理 1 可知,当取 $\hat{\alpha}_1 = \hat{\alpha}_2\sqrt{\frac{\beta\tau_1}{2n}}$,$\hat{\alpha}_2 = 1$ 时有 $(x(\alpha),y(\alpha),s(\alpha)) \in N(\tau_1,\beta)$. 由(18)式得

$$\begin{aligned}\mu(\hat{\alpha}) &\leqslant (1 - \hat{\alpha}_1(1-\tau_1) + \hat{\alpha}_2\beta\tau_1/\sqrt{n})\mu\\ &= (1 - \hat{\alpha}_2(1 - \tau_1 - \sqrt{2\beta\tau_1})\sqrt{\beta\tau_1/(2n)})\mu\\ &\leqslant \left(1 - \left(1 - \frac{1}{4} - \frac{1}{2}\right)\sqrt{\beta\tau_1/(2n)}\right)\mu\\ &\leqslant \left(1 - \frac{\sqrt{\beta\tau_1}}{4\sqrt{2n}}\right)\mu. \qquad (20)\end{aligned}$$

因此,由搜索(7)得到的迭代点满足 $\mu_{k+1} \leqslant \left(1 - \frac{\sqrt{\beta\tau_1}}{4\sqrt{2n}}\right)\mu_k$,$k = 0,1,2,\cdots$. 故当 $k \geqslant \frac{4\sqrt{2n}}{\sqrt{\beta\tau_1}}\log\frac{(x^0)^Ts^0}{\varepsilon}$ 时有

$$\begin{aligned}(x^k)^Ts^k &\leqslant \left(1 - \frac{\sqrt{\beta\tau_1}}{4\sqrt{2n}}\right)^k(x^0)^Ts^0\\ &= (x^0)^Ts^0e^{k\log\left(1-\frac{\sqrt{\beta\tau_1}}{4\sqrt{2n}}\right)}\\ &\leqslant (x^0)^Ts^0e^{-\log\frac{(x^0)^Ts^0}{\varepsilon}} = \varepsilon.\end{aligned}$$

所以,算法最多需要 $\lceil\frac{4\sqrt{2n}}{\sqrt{\beta\tau_1}}\log\frac{(x^0)^Ts^0}{\varepsilon}\rceil$ 次迭代后终止.

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

由表达式(18)可以看出,对于固定 $\alpha_2 \in (0,1]$,对偶间隙关于 α_1 单调下降,因此可以用下面方法计算迭代步长: 首先取 $\alpha_2 = \theta \in (0,1]$,然后求满足 $(x(\alpha), y(\alpha), s(\alpha)) \in N(\tau_1, \beta)$ 的最大值 $\alpha_1 \in (0,1]$. 由定理1可知,必定有 $\alpha_1 \geqslant \theta\sqrt{\beta\tau_1/(2n)}$. 因此,由复杂性证明中的(20)式可知,若在算法1中采用上面方法计算迭代步长,则其迭代复杂性仍然是 $O\left(\sqrt{n}\log\frac{(x^0)^T s^0}{\varepsilon}\right)$.

参 考 文 献

[1] Peng J M, Terlaky T, Zhao Y B. A predictor-corrector algorithm for linear optimization based on a specific self-regular proximity function [J]. SIAM J Optim, 2005, 15: 1105 - 1127.

[2] Ai W, Zhang S. An $O(\sqrt{n}L)$ -iteration primal-dual path-following method, based on wide neighborhoods and large updates, for monotone LCP [J]. SIAM J Optim, 2005, 16: 400 - 417.

[3] Mehrotra S. On the implementation of a primal-dual interior point method [J]. SIAM J Optim, 1992, 2: 575 - 601.

[4] Salahi M, Peng J, Terlaky T. On Mehrotra-type predictor-corrector algorithms [J]. SIAM J Optim, 2007, 18: 1377 - 1397.

[5] Liu C, Liu H, Cong W. An $O(\sqrt{n}L)$ iteration primal-dual second-order corrector algorithm for linear programming [J]. Optim Lett. 2011, 5: 729 - 743.

A New Mehrotra-Type Predictor-Corrector Algorithm for Linear Programming

Liu Changhe　Wu Dan

(School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang　471023)

Abstract

In interior point methods (IPMs) for linear programming, there has been an inconsistency between theory and practice: fast algorithms in practice may actually render worse complexity bounds. The majority of commercial and public IPM codes implement a variant of Mehrotra's predictor - corrector algorithm. We propose a new Mehrotra-type predictor-corrector algorithm. The $O(\sqrt{n}L)$ iteration complexity of the algorithm is given, the same as the best complexity result for IPMs.

Key words: linear programming, interior point methods, Mehrotra-type predictor-corrector algorithm, polynomial complexity.

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net