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$C^3 \otimes C^d$系统中的纠缠见证优化*

赵　慧**,杨　玉
(北京工业大学理学部,北京　100124)

摘要:利用纠缠见证的定义和算子半正定的充要条件,得到$C^3 \otimes C^d$量子系统中厄米算子($\boldsymbol{W}$)是纠缠见证的充要条件.把$C^3 \otimes C^d$中$\boldsymbol{W}$表示为分块矩阵,利用子块之间的关系,得到对任意可分态($\boldsymbol{\rho}_{sep}$),$\boldsymbol{W}$是正算子的充要条件.对纠缠见证进行非线性改进,得到仅对部分可观测量进行测量,便能判别量子态是否纠缠的条件,并用例子验证,非线性改进后的纠缠见证能够判别出更多的纠缠态,优于其他文献中的纠缠见证.

关键词:纠缠见证;可分;纠缠;密度矩阵;混合态;半正定
中图分类号:O413

Entanglement witness optimization in $C^3 \otimes C^d$ system*

ZHAO Hui**, YANG Yu
(Faculty of Science, Beijing University of Technology, Beijing　100124)

Abstract: Using the definition of entanglement witness and the necessary and sufficient condition of positive semi-definite operator, we present the necessary and sufficient condition for the Hermitian operator ($\boldsymbol{W}$) to be the entanglement witness in the $C^3 \otimes C^d$ quantum systems. The Hermitian operator in $C^3 \otimes C^d$ is represented as a block matrix, using the relationship of the sub-blocks, the necessary and sufficient conditions of $\boldsymbol{W}$ to be a positive operator is obtained for any separable state ($\boldsymbol{\rho}_{sep}$). Then the entanglement witness is improved non-linearly, the conditions for determining whether the quantum state is entangled or not by measuring only a part of the observable is obtained. Moreover, use examples to verify that the nonlinearly improved entanglement witness can detect more entangled states, which is better than the entanglement witness in other studies.

Keywords: entanglement witness; separability; entanglement; density matrix; mixed state; positive semidefinite
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0　引　言

量子信息是近年来发展非常迅速的多学科交叉研究领域,量子纠缠是量子信息中重要的研究内容.1997年,Horodecki[1]给出了束缚纠缠态的例子,并在量子系统上构造了一类positive partial transposition(PPT)纠缠态;Zou等[2]讨论了非马尔可夫效应和失谐对量子熵不确定关系和纠缠见证下界的

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** 通信作者:zhaohui@bjut.edu.cn

影响；2015年，Saideh等[3]利用PPT准则揭示了量子相关性的存在，并将基于方差的纠缠见证推广到高维量子系统；Tan等[4]构造了多体量子系统中的纠缠见证；Bruns等[5]应用Choi-Jamiolkowski同构，揭示了两体量子态的纠缠特性；Coto等[6]利用基态生成了多体最大纠缠态；Cruz[7]给出了纠缠见证、纠缠生成、贝尔不等式违背和几何量子失谐的分析解释；Bartkiewicz等[8]分析了任意双量子比特的贝尔非局域性和纠缠见证的测量度；Kühn等[9]推导出验证非高斯纠缠准则，且构造方法能够验证复杂光状态的两体和多体纠缠；Deb和Ghosh[10]利用并发度和纠缠见证刻画纠缠；2017年，Oudot等[11]揭示了空间分裂自旋压缩在玻色-爱因斯坦凝聚态中的关系，并由此得到了最优的纠缠见证；Akbari-Kourbolagh和Azhdargalam[12]构造了三量子比特的纠缠见证，且这些见证能够更有效地检测束缚纠缠态；Li等[13]利用互无偏测量构造了纠缠见证；Amaro和Müller[14]基于图态和稳定态之间的局域幺正等价，构造了局域纠缠见证；Zhao等[15]研究了多体量子态的纠缠性质；Shen等[16]证明了对$C^2 \otimes C^d$上任意一个纠缠见证都可以用非线性方法进行改进，且改进后的纠缠见证优于原来的纠缠见证.纠缠见证为判别量子态是否纠缠提供了一个重要的工具，对纠缠见证进行优化是人们日益关注的研究热点.

本文借助了新的原理和方法，即对$C^3 \otimes C^d$系统中的纠缠见证进行非线性改进，并用实例证明了非线性改进后的纠缠见证可以判别更多的纠缠态，得到优化后的纠缠见证在量子态纠缠判别上具有较好优越性的结论.

1　基础知识

定义1　量子系统的态为$|\phi_i\rangle$，则称$\{p_i,|\phi_i\rangle\}$是态的系综.式中i为指标，p_i为对应的概率.系统的密度矩阵定义为$\rho=\sum_i p_i|\phi_i\rangle\langle\phi_i|$. 若$\rho=|\phi\rangle\langle\phi|$，$\mathrm{Tr}(\rho^2)=1$，则称$\rho$是纯态；若$\mathrm{Tr}(\rho^2)<1$，则称$\rho$是混态.

定义2　一个矩阵ρ是和某个系综$\{p_i,|\phi_i\rangle\}$相关联的密度矩阵，当且仅当满足以下条件：

(i)迹条件：ρ的迹是1，即$\mathrm{Tr}(\rho)=1$；

(ii)半正定条件：ρ是一个半正定矩阵.

定义3　从$V\times V$到复空间C的函数$(\cdot,\cdot)$是内积，如果满足以下条件：

(i)$(\cdot,\cdot)$对第2个自变量是线性的，即$\left(|v\rangle,\sum_i\lambda_i|w_i\rangle\right)=\sum_i\lambda_i\left(|v\rangle,|w_i\rangle\right)$；

(ii)$\left(|v\rangle,|w\rangle\right)=\left(|w\rangle,|v\rangle\right)^*$，其中*表示复共轭；

(iii)$\left(|v\rangle,|w\rangle\right)\geqslant 0$，当且仅当$|v\rangle=0$时取等号.

定义4　假设W是作用在复空间$C^m\otimes C^d$上的一个厄米算子，W是一个纠缠见证，如果满足以下条件[17]：

(i)W是分块正的，即对所有可分态，有$\mathrm{Tr}(W\rho_{\mathrm{sep}})\geqslant 0$；

(ii)在$C^m\otimes C^d$中至少存在一个纠缠态ρ_{ent}使得$\mathrm{Tr}(W\rho_{\mathrm{ent}})<0$.

2　$C^3\otimes C^d$上纠缠见证的非线性改进

本文考虑作用在$C^3\otimes C^d$上的一个W，

$$W=\begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}^{\dagger} & W_{22} & W_{23} \\ W_{13}^{\dagger} & W_{23}^{\dagger} & W_{33} \end{pmatrix}, \tag{1}$$

式中：$W_{ii}(i=1,2,3)$和$W_{ij}(i=1,2,j=2,3,i\neq j)\in C^{d\times d}$，$\dagger$表示对矩阵进行共轭转置.众所周知，对$C^d$中的任意纯态$|\xi\rangle$，$\left(\langle\xi|W|\xi\rangle\right)\geqslant 0$当且仅当$\left(\langle\xi|W|\xi\rangle\right)$的所有主子式都≥0.

根据纠缠见证的定义，可以得到$C^3\otimes C^d$系统上的W是纠缠见证的充要条件.

定理1　设W形如式(1)，W是一个纠缠见证的充要条件是：

(i)W至少存在一个负特征值；

(ii)$W_{11}\geqslant 0$，$W_{22}\geqslant 0$，$W_{33}\geqslant 0$，

$$\langle\xi|W_{11}|\xi\rangle\langle\xi|W_{22}|\xi\rangle\geqslant\left|\langle\xi|W_{12}|\xi\rangle\right|^2, \tag{2}$$

$$\langle\xi|W_{11}|\xi\rangle\langle\xi|W_{33}|\xi\rangle\geqslant\left|\langle\xi|W_{13}|\xi\rangle\right|^2, \tag{3}$$

$$\langle\xi|W_{22}|\xi\rangle\langle\xi|W_{33}|\xi\rangle\geqslant\left|\langle\xi|W_{23}|\xi\rangle\right|^2, \tag{4}$$

$$\langle\xi|W_{11}|\xi\rangle\langle\xi|W_{22}|\xi\rangle\langle\xi|W_{33}|\xi\rangle\geqslant$$
$$\langle\xi|W_{11}|\xi\rangle\left|\langle\xi|W_{23}|\xi\rangle\right|^2+\langle\xi|W_{22}|\xi\rangle\left|\langle\xi|W_{13}|\xi\rangle\right|^2+$$
$$\langle\xi|W_{33}|\xi\rangle\left|\langle\xi|W_{12}|\xi\rangle\right|^2-$$
$$2\mathrm{Re}\left(\langle\xi|W_{12}|\xi\rangle\langle\xi|W_{13}^{\dagger}|\xi\rangle\langle\xi|W_{23}|\xi\rangle\right), \tag{5}$$

式中：$|\xi\rangle$是C^d中的任意纯态.

证明　因为W至少有一个负特征值，那么对于

量子态$\boldsymbol{\rho}$,有$\mathrm{Tr}\left(\boldsymbol{W\rho}\right)<0$,所以$\boldsymbol{\rho}$是纠缠的,即"$\boldsymbol{W}$至少存在一个负特征值"和定义4中的(ii)等价.只需证明式(2)~(5)和定义4中的(i)等价.定义4中的(i)成立当且仅当对任意纯态$|\boldsymbol{\phi}\rangle\in C^3$和$|\boldsymbol{\xi}\rangle\in C^d$,

$$\mathrm{Tr}\left(\boldsymbol{W}\left|\boldsymbol{\phi}\right\rangle\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\left\langle\boldsymbol{\phi}\right|\right)=$$

$$\left\langle\boldsymbol{\phi},\boldsymbol{\xi}\right|\boldsymbol{W}\left|\boldsymbol{\phi},\boldsymbol{\xi}\right\rangle=\left\langle\boldsymbol{\phi}\right|\left(\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}\left|\boldsymbol{\xi}\right\rangle\right)\left|\boldsymbol{\phi}\right\rangle=$$

$$\left\langle\boldsymbol{\phi}\right|\begin{pmatrix}\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{11}\left|\boldsymbol{\xi}\right\rangle & \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{12}\left|\boldsymbol{\xi}\right\rangle & \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{13}\left|\boldsymbol{\xi}\right\rangle\\ \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{12}^{\dagger}\left|\boldsymbol{\xi}\right\rangle & \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{22}\left|\boldsymbol{\xi}\right\rangle & \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{23}\left|\boldsymbol{\xi}\right\rangle\\ \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{13}^{\dagger}\left|\boldsymbol{\xi}\right\rangle & \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{23}^{\dagger}\left|\boldsymbol{\xi}\right\rangle & \left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{33}\left|\boldsymbol{\xi}\right\rangle\end{pmatrix}\left|\boldsymbol{\phi}\right\rangle\geqslant 0. \tag{6}$$

$\mathrm{Tr}\left(\boldsymbol{W}\left|\boldsymbol{\phi}\right\rangle\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\left\langle\boldsymbol{\phi}\right|\right)\geqslant 0$,可得对任意纯态$|\boldsymbol{\xi}\rangle\in\boldsymbol{C}^d$,$\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}\left|\boldsymbol{\xi}\right\rangle\geqslant 0$.而$\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}\left|\boldsymbol{\xi}\right\rangle\geqslant 0$等价于对任意纯态$|\boldsymbol{\xi}\rangle\in\boldsymbol{C}^d$,其所有主子式都≥0,即式(2)~(5)成立[18].

设$\boldsymbol{W}$定义形如式(1),令

$$\tilde{\boldsymbol{W}}_{11}=\begin{pmatrix}\boldsymbol{W}_{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\tilde{\boldsymbol{W}}_{22}=\begin{pmatrix}0 & 0 & 0\\ 0 & \boldsymbol{W}_{22} & 0\\ 0 & 0 & 0\end{pmatrix},\tilde{\boldsymbol{W}}_{33}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \boldsymbol{W}_{33}\end{pmatrix},$$

$$\tilde{\boldsymbol{W}}_{12}=\begin{pmatrix}0 & \boldsymbol{W}_{12} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\tilde{\boldsymbol{W}}_{13}=\begin{pmatrix}0 & 0 & \boldsymbol{W}_{13}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\tilde{\boldsymbol{W}}_{23}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \boldsymbol{W}_{23}\\ 0 & 0 & 0\end{pmatrix}. \tag{7}$$

考虑可分纯态$\boldsymbol{\rho}_{\mathrm{sep}}=\left|\boldsymbol{\eta}\right\rangle\left\langle\boldsymbol{\eta}\right|$,其中

$$|\boldsymbol{\eta}\rangle=|\boldsymbol{\phi}\rangle\otimes|\boldsymbol{\xi}\rangle,|\boldsymbol{\phi}\rangle=\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}. \tag{8}$$

经计算可得:

当$i=1,2,3$时,

$$\left\langle\tilde{\boldsymbol{W}}_{ii}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}=\left|a_i\right|^2\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ii}\left|\boldsymbol{\xi}\right\rangle. \tag{9}$$

当$i=1,2,j=2,3,i\neq j$时,

$$\left\langle\tilde{\boldsymbol{W}}_{ij}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}=a_i^{*}a_j\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ij}\left|\boldsymbol{\xi}\right\rangle, \tag{10}$$

$$\left\langle\tilde{\boldsymbol{W}}_{ij}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}+\left\langle\tilde{\boldsymbol{W}}_{ij}^{\dagger}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}=2\mathrm{Re}\left(a_i^{*}a_j\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ij}\left|\boldsymbol{\xi}\right\rangle\right). \tag{11}$$

当$k=1,2,3,i=1,2,j=2,3,i\neq j\neq k,i<j$时,

$$\left\langle\tilde{\boldsymbol{W}}_{kk}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\left\langle\tilde{\boldsymbol{W}}_{ij}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\left\langle\tilde{\boldsymbol{W}}_{ij}^{\dagger}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}=$$

$$\left|a_1\right|^2\left|a_2\right|^2\left|a_3\right|^2\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{kk}\left|\boldsymbol{\xi}\right\rangle\left|\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ij}\left|\boldsymbol{\xi}\right\rangle\right|^2, \tag{12}$$

$$\left\langle\tilde{\boldsymbol{W}}_{12}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\left\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\left\langle\tilde{\boldsymbol{W}}_{23}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}=$$

$$\left|a_1\right|^2\left|a_2\right|^2\left|a_3\right|^2\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{12}\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{13}^{\dagger}\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{23}\left|\boldsymbol{\xi}\right\rangle. \tag{13}$$

下面给出$\boldsymbol{W}$分块正定的充要条件.

定理2　$\boldsymbol{W}$是分块正定的,当且仅当对任意可分态$\boldsymbol{\rho}_{\mathrm{sep}}\in C^3\otimes C^d$,有$\boldsymbol{W}_{11}\geqslant 0,\boldsymbol{W}_{22}\geqslant 0,\boldsymbol{W}_{33}\geqslant 0$,

$$\left\langle\tilde{\boldsymbol{W}}_{11}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{22}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\geqslant\frac{1}{4}\left|\left\langle\tilde{\boldsymbol{W}}_{12}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\left\langle\tilde{\boldsymbol{W}}_{12}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right|^2, \tag{14}$$

$$\left\langle\tilde{\boldsymbol{W}}_{11}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{33}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\geqslant\frac{1}{4}\left|\left\langle\tilde{\boldsymbol{W}}_{13}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\left\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right|^2, \tag{15}$$

$$\left\langle\tilde{\boldsymbol{W}}_{22}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{33}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\geqslant\frac{1}{4}\left|\left\langle\tilde{\boldsymbol{W}}_{23}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\left\langle\tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right|^2, \tag{16}$$

$$\left\langle\tilde{\boldsymbol{W}}_{11}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{22}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{33}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\geqslant$$

$$\left\langle\tilde{\boldsymbol{W}}_{11}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{23}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+$$

$$\left\langle\tilde{\boldsymbol{W}}_{22}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{13}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\left\langle\tilde{\boldsymbol{W}}_{33}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{12}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{12}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}-$$

$$2\mathrm{Re}\left(\left\langle\tilde{\boldsymbol{W}}_{12}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\left\langle\tilde{\boldsymbol{W}}_{23}\right\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right). \tag{17}$$

证明　由定理1,$\boldsymbol{W}$是分块正定的,当且仅当对任意的$|\boldsymbol{\xi}\rangle\in C^d$,式(2)~(5)都成立.因此,只需证明式(2)~(5)和式(14)~(17)是等价的.

(i)式(14)~(17)⇒式(2)~(5).

对任意纯可分态$\left|\boldsymbol{\eta}\right\rangle$形如式(8),由式(14)~(17),当$i=1,2,j=2,3,i\neq j$时,可得

$$\left\langle\tilde{\boldsymbol{W}}_{ii}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\left\langle\tilde{\boldsymbol{W}}_{jj}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}=$$

$$\left|a_i\right|^2\left|a_j\right|^2\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ii}\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{jj}\left|\boldsymbol{\xi}\right\rangle\geqslant$$

$$\frac{1}{4}\left|\left\langle\tilde{\boldsymbol{W}}_{ij}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}+\left\langle\tilde{\boldsymbol{W}}_{ij}^{\dagger}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\right|^2=$$

$$\left|\mathrm{Re}\left(a_i^{*}a_j\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ij}\left|\boldsymbol{\xi}\right\rangle\right)\right|^2, \tag{18}$$

$$\left\langle\tilde{\boldsymbol{W}}_{11}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\left\langle\tilde{\boldsymbol{W}}_{22}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}\left\langle\tilde{\boldsymbol{W}}_{33}\right\rangle_{|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|}=$$

$$\left|a_1\right|^2\left|a_2\right|^2\left|a_3\right|^2\left(\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{11}\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{22}\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{33}\left|\boldsymbol{\xi}\right\rangle\right)\geqslant$$

$$\left|a_1\right|^2\left|a_2\right|^2\left|a_3\right|^2\left(\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{11}\left|\boldsymbol{\xi}\right\rangle\left|\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{23}\left|\boldsymbol{\xi}\right\rangle\right|^2+\right.$$

$$\left.\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{22}\left|\boldsymbol{\xi}\right\rangle\left|\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{13}\left|\boldsymbol{\xi}\right\rangle\right|^2+\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{33}\left|\boldsymbol{\xi}\right\rangle\left|\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{12}\left|\boldsymbol{\xi}\right\rangle\right|^2\right)-$$

$$\left|a_1\right|^2\left|a_2\right|^2\left|a_3\right|^2\cdot 2\mathrm{Re}\left(\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{12}\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{13}^{\dagger}\left|\boldsymbol{\xi}\right\rangle\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{23}\left|\boldsymbol{\xi}\right\rangle\right). \tag{19}$$

假设$\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ij}\left|\boldsymbol{\xi}\right\rangle=x+\sqrt{-1}\,y,x,y\in R$,那么,$\left\langle\boldsymbol{\xi}\right|\boldsymbol{W}_{ij}\left|\boldsymbol{\xi}\right\rangle=\sqrt{x^2+y^2}\left(\cos\theta+\sqrt{-1}\sin\theta\right)$,其中$-\pi<\theta\leqslant\pi$.

对不等式(18),取$a_i=\frac{1}{\sqrt{2}},a_j=\frac{1}{\sqrt{2}}(\cos\theta-$

$\sqrt{-1}\sin\theta$),则有

$$\frac{1}{4}\langle\xi|\boldsymbol{W}_{ii}|\xi\rangle\langle\xi|\boldsymbol{W}_{jj}|\xi\rangle \geqslant \frac{1}{4}\left|\langle\tilde{\boldsymbol{W}}_{ij}\rangle_{|\eta\rangle\langle\eta|}+\langle\tilde{\boldsymbol{W}}_{ij}^{\dagger}\rangle_{|\eta\rangle\langle\eta|}\right|^2 =$$

$$\left|\mathrm{Re}\left(a_i^* a_j\langle\xi|\boldsymbol{W}_{ij}|\xi\rangle\right)\right|^2 = \frac{1}{4}(x^2+y^2) =$$

$$\frac{1}{4}\left|\langle\xi|\boldsymbol{W}_{ij}|\xi\rangle>\right|^2.$$

可得式(2)~(4)成立.

对不等式(19),取$a_1=\frac{1}{\sqrt{3}}$, $a_2=\frac{1}{\sqrt{3}}$, $a_3=\frac{1}{\sqrt{3}}\left(\cos\theta-\sqrt{-1}\sin\theta\right)$,则有

$$\frac{1}{27}\langle\xi|\boldsymbol{W}_{11}|\xi\rangle\langle\xi|\boldsymbol{W}_{22}|\xi\rangle\langle\xi|\boldsymbol{W}_{33}|\xi\rangle \geqslant$$

$$\frac{1}{27}\left(\langle\xi|\boldsymbol{W}_{11}|\xi\rangle\left|\langle\xi|\boldsymbol{W}_{23}|\xi\rangle\right|^2+\right.$$

$$\left.\langle\xi|\boldsymbol{W}_{22}|\xi\rangle\left|\langle\xi|\boldsymbol{W}_{13}|\xi\rangle\right|^2+\langle\xi|\boldsymbol{W}_{33}|\xi\rangle\left|\langle\xi|\boldsymbol{W}_{12}|\xi\rangle\right|^2\right)-$$

$$\frac{1}{27}\cdot 2\mathrm{Re}\left(\langle\xi|\boldsymbol{W}_{12}|\xi\rangle\langle\xi|\boldsymbol{W}_{13}^{\dagger}|\xi\rangle\langle\xi|\boldsymbol{W}_{23}|\xi\rangle\right).$$

可得式(5)成立.

(ii) 式(2)~(5)⇒式(14)~(17).

设$a_2=a+\sqrt{-1}\,b$, $a_3=c+\sqrt{-1}\,d$, $\langle\xi|\boldsymbol{W}_{23}|\xi\rangle=e+\sqrt{-1}\,f$,经计算得

$$|a_2|^2|a_3|^2\left|\langle\xi|\boldsymbol{W}_{23}|\xi\rangle\right|^2-\left|\mathrm{Re}\left(a_2^* a_3\langle\xi|\boldsymbol{W}_{23}|\xi\rangle\right)\right|^2 =$$

$$\left[(ad-bc)e+(ac+bd)f\right]^2 \geqslant 0.$$

对任意形如式(8)的可分纯态$\boldsymbol{\rho}_{\mathrm{sep}}$,利用式(9)和(11)得

$$\langle\tilde{\boldsymbol{W}}_{22}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{33}\rangle_{|\eta\rangle\langle\eta|} =$$

$$|a_2|^2|a_3|^2\langle\xi|\boldsymbol{W}_{22}|\xi\rangle\langle\xi|\boldsymbol{W}_{33}|\xi\rangle \geqslant$$

$$|a_2|^2|a_3|^2\left|\langle\xi|\boldsymbol{W}_{23}|\xi\rangle\right|^2 \geqslant \left|\mathrm{Re}\left(a_2^* a_3\langle\xi|\boldsymbol{W}_{23}|\xi\rangle\right)\right|^2 =$$

$$\frac{1}{4}\left|\langle\tilde{\boldsymbol{W}}_{23}\rangle_{|\eta\rangle\langle\eta|}+\langle\tilde{\boldsymbol{W}}_{23}^{\dagger}\rangle_{|\eta\rangle\langle\eta|}\right|^2.$$

同理,可得

$$\langle\tilde{\boldsymbol{W}}_{ii}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{jj}\rangle_{|\eta\rangle\langle\eta|} \geqslant \frac{1}{4}\left|\langle\tilde{\boldsymbol{W}}_{ij}\rangle_{|\eta\rangle\langle\eta|}+\langle\tilde{\boldsymbol{W}}_{ij}^{\dagger}\rangle_{|\eta\rangle\langle\eta|}\right|^2,$$

$$i=1,2, j=2,3, i\neq j. \tag{20}$$

由式(5)、(9)和(10)得

$$\langle\tilde{\boldsymbol{W}}_{11}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{22}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{33}\rangle_{|\eta\rangle\langle\eta|} \geqslant$$

$$\langle\tilde{\boldsymbol{W}}_{11}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{23}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{23}^{\dagger}\rangle_{|\eta\rangle\langle\eta|}+$$

$$\langle\tilde{\boldsymbol{W}}_{22}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{13}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\rangle_{|\eta\rangle\langle\eta|}+$$

$$\langle\tilde{\boldsymbol{W}}_{33}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{12}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{12}^{\dagger}\rangle_{|\eta\rangle\langle\eta|}-$$

$$2\mathrm{Re}\left(\langle\tilde{\boldsymbol{W}}_{12}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\rangle_{|\eta\rangle\langle\eta|}\langle\tilde{\boldsymbol{W}}_{23}\rangle_{|\eta\rangle\langle\eta|}\right).$$

对任意可分混态$\boldsymbol{\rho}_{\mathrm{sep}}=\sum_i p_i|\boldsymbol{\tau}_i\rangle\langle\boldsymbol{\tau}_i|$,其中$|\boldsymbol{\tau}_i\rangle=|\boldsymbol{\phi}_i\rangle\otimes|\boldsymbol{\xi}_i\rangle$, $p_i>0$, $\sum_i p_i=1$,设

$$\boldsymbol{A}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}+\tilde{\boldsymbol{W}}_{22}\right), \boldsymbol{B}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}-\tilde{\boldsymbol{W}}_{22}\right),$$

$$\boldsymbol{C}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}+\tilde{\boldsymbol{W}}_{33}\right),$$

$$\boldsymbol{D}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}-\tilde{\boldsymbol{W}}_{33}\right), \boldsymbol{E}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{22}+\tilde{\boldsymbol{W}}_{33}\right),$$

$$\boldsymbol{F}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{22}-\tilde{\boldsymbol{W}}_{33}\right).$$

由式(18)、(19)和$a_i^2\geqslant b_i^2+c_i^2$, $\left(\sum_i^n p_i a_i\right)^2\geqslant\left(\sum_i^n p_i b_i\right)^2+\left(\sum_i^n p_i c_i\right)^2$,其中$b_i$、$c_i$是任意实数,$a_i$是非负的[19],可得

$$\langle\boldsymbol{A}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2 \geqslant \langle\boldsymbol{B}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2+\frac{1}{4}\left|\langle\tilde{\boldsymbol{W}}_{12}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\langle\tilde{\boldsymbol{W}}_{12}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right|^2, \tag{21}$$

$$\langle\boldsymbol{C}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2 \geqslant \langle\boldsymbol{D}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2+\frac{1}{4}\left|\langle\tilde{\boldsymbol{W}}_{13}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right|^2, \tag{22}$$

$$\langle\boldsymbol{E}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2 \geqslant \langle\boldsymbol{F}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2+\frac{1}{4}\left|\langle\tilde{\boldsymbol{W}}_{23}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\langle\tilde{\boldsymbol{W}}_{23}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right|^2, \tag{23}$$

$$\langle\boldsymbol{A}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2\langle\boldsymbol{C}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\langle\boldsymbol{B}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2\langle\boldsymbol{D}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}} \geqslant$$

$$\langle\boldsymbol{A}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2\langle\boldsymbol{D}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\langle\boldsymbol{B}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}^2\langle\boldsymbol{C}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+$$

$$\langle\tilde{\boldsymbol{W}}_{11}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{23}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{23}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+$$

$$\langle\tilde{\boldsymbol{W}}_{22}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{13}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+$$

$$\langle\tilde{\boldsymbol{W}}_{33}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{12}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{12}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}-$$

$$2\mathrm{Re}\left(\langle\tilde{\boldsymbol{W}}_{12}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{13}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{23}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right). \tag{24}$$

因式(21)~(24)和式(14)~(17)是等价的,所以由式(2)~(5)可得式(14)~(17).

推论1　$\boldsymbol{W}$是一个纠缠见证,$\boldsymbol{W}$定义形如式(1),那么对$C^3\otimes C^d$中任意可分态$\boldsymbol{\rho}_{\mathrm{sep}}$满足如下不等式:

$$\langle\tilde{\boldsymbol{W}}_{11}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\langle\tilde{\boldsymbol{W}}_{22}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}} \geqslant \frac{1}{4}\left|\langle\tilde{\boldsymbol{W}}_{12}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}+\langle\tilde{\boldsymbol{W}}_{12}^{\dagger}\rangle_{\boldsymbol{\rho}_{\mathrm{sep}}}\right|, \tag{25}$$

$$\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho_{\mathrm{sep}}}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho_{\mathrm{sep}}}\geqslant\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{13}\right\rangle_{\rho_{\mathrm{sep}}}+\left\langle \tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\rho_{\mathrm{sep}}}\right|,\tag{26}$$

$$\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho_{\mathrm{sep}}}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho_{\mathrm{sep}}}\geqslant\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{\rho_{\mathrm{sep}}}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\rho_{\mathrm{sep}}}\right|.\tag{27}$$

用纠缠见证 $\boldsymbol{W}$ 去判别纠缠,需要测量9个可观测量,但是用式(25)~(27)去判别纠缠,只需要对式(7)中的 $\tilde{\boldsymbol{W}}_{11}$、$\tilde{\boldsymbol{W}}_{22}$ 和 $\tilde{\boldsymbol{W}}_{33}$ 这3个可观测量进行测量.因此,与原有的纠缠见证 $\boldsymbol{W}$ 相比,式(25)~(27)计算更简便.

定理3　式(25)~(27)可以判别满足不等式 $\langle\boldsymbol{W}\rangle_{\rho}<0$ 的任意纠缠态 $\boldsymbol{\rho}$,其中 $\boldsymbol{W}$ 是定义在式(1)中的一个纠缠见证.

证明　设 $\boldsymbol{\rho}$ 是 $C^3\otimes C^d$ 中的任意一个态,则

$$\left\langle \tilde{\boldsymbol{W}}_{ii}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{jj}\right\rangle_{\rho}\geqslant\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{ij}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{ij}^{\dagger}\right\rangle_{\rho}\right|^2,$$

$$i=1,2,j=2,3,i\neq j.\tag{28}$$

可得

$$\langle\boldsymbol{W}\rangle_{\rho}=\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{12}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{12}^{\dagger}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{13}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\rho}=$$

$$\frac{1}{2}\left(\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho}\right)+\frac{1}{2}\left(\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho}\right)+\frac{1}{2}\left(\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho}\right)+\left\langle \tilde{\boldsymbol{W}}_{12}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{12}^{\dagger}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{13}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\rho}\geqslant$$

$$\sqrt{\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho}}+\sqrt{\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho}}+\sqrt{\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho}}+2\sqrt{\left\langle \tilde{\boldsymbol{W}}_{12}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{12}^{\dagger}\right\rangle_{\rho}}+2\sqrt{\left\langle \tilde{\boldsymbol{W}}_{13}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\rho}}+2\sqrt{\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\rho}}\geqslant 0.$$

对任意纠缠态 $\boldsymbol{\rho}$,若满足不等式条件 $\langle\boldsymbol{W}\rangle_{\rho}<0$,则有 $\left\langle \tilde{\boldsymbol{W}}_{ii}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{jj}\right\rangle_{\rho}<\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{ij}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{ij}^{\dagger}\right\rangle_{\rho}\right|^2$,与式(28)相悖.因此,$\boldsymbol{\rho}$ 可被式(25)~(27)判别.

引理1　若 $\boldsymbol{W}_{22}>0$,那么 $\boldsymbol{W}\geqslant 0$,可得 $\boldsymbol{W}_{33}-\boldsymbol{W}_{23}^{\dagger}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\geqslant 0$.

证明:$\boldsymbol{W}_{22}>0\Rightarrow\boldsymbol{W}_{22}$ 可逆.

设 $\boldsymbol{T}=\begin{pmatrix}E & 0 & 0\\ 0 & E & 0\\ 0 & -\boldsymbol{W}_{23}^{\dagger}\boldsymbol{W}_{22}^{-1} & E\end{pmatrix}$,$\boldsymbol{T}$ 可逆.

则

$$\boldsymbol{T}\boldsymbol{W}\boldsymbol{T}^{\dagger}=\begin{pmatrix}\boldsymbol{W}_{11} & \boldsymbol{W}_{12} & \boldsymbol{W}_{13}-\boldsymbol{W}_{12}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\\ \boldsymbol{W}_{12}^{\dagger} & \boldsymbol{W}_{22} & \boldsymbol{0}\\ \left(\boldsymbol{W}_{13}-\boldsymbol{W}_{12}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\right)^{\dagger} & \boldsymbol{0} & \boldsymbol{W}_{33}-\boldsymbol{W}_{23}^{\dagger}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\end{pmatrix}=\boldsymbol{M}.$$

因此,$\boldsymbol{W}$ 和 $\boldsymbol{M}$ 是酉合同的.

$\boldsymbol{W}\geqslant 0\Rightarrow\boldsymbol{M}\geqslant 0\Rightarrow\boldsymbol{M}$ 的所有主子式都≥0 $\Rightarrow\boldsymbol{W}_{33}-\boldsymbol{W}_{23}^{\dagger}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\geqslant 0$.

同理,可得

$\boldsymbol{W}_{11}>0$,那么 $\boldsymbol{W}\geqslant 0\Rightarrow\boldsymbol{W}_{33}-\boldsymbol{W}_{13}^{\dagger}\boldsymbol{W}_{11}^{-1}\boldsymbol{W}_{13}\geqslant 0$.

同理,当

$$\boldsymbol{W}_{11}>0,$$

那么 $\boldsymbol{W}\geqslant 0\Rightarrow\boldsymbol{W}_{22}-\boldsymbol{W}_{12}^{\dagger}\boldsymbol{W}_{11}^{-1}\boldsymbol{W}_{12}\geqslant 0$.

定理4　设 $\boldsymbol{W}_{11}\geqslant 0,\boldsymbol{W}_{22}\geqslant 0,\boldsymbol{W}_{33}\geqslant 0,\tilde{\boldsymbol{W}}_{ii},i=1,2,3;\tilde{\boldsymbol{W}}_{ij},i=1,2,j=2,3,i\neq j$ 的定义形如式(7).那么 $\boldsymbol{W}\geqslant 0$ 当且仅当对任意态 $\boldsymbol{\rho}\in C^3\otimes C^d$,下式成立:

$$\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho}\geqslant\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{12}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{12}^{\dagger}\right\rangle_{\rho}\right|^2,\tag{29}$$

$$\left\langle \tilde{\boldsymbol{W}}_{11}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho}\geqslant\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{13}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\rho}\right|^2,\tag{30}$$

$$\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{\rho}\geqslant\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\rho}\right|^2.\tag{31}$$

证明　式(29)~(31)$\Rightarrow\boldsymbol{W}\geqslant 0$:

对任意态 $\boldsymbol{\rho}\in C^3\otimes C^d$,由定理3的证明过程和式(29)~(31),可推得 $\langle\boldsymbol{W}\rangle_{\rho}\geqslant 0$,那么 $\boldsymbol{W}\geqslant 0$.

$\boldsymbol{W}\geqslant 0\Rightarrow$ 式(29)~(31).

若 $\boldsymbol{W}_{22}>0$,

对任意纯态 $|\boldsymbol{\phi}\rangle\in C^3\otimes C^d$,$|\boldsymbol{\phi}\rangle=\begin{pmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\\ \boldsymbol{\phi}_3\end{pmatrix}$,$|\boldsymbol{\phi}_i\rangle\in C^d$,

由式(11)可得

$$\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{|\phi\rangle\langle\phi|}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{|\phi\rangle\langle\phi|}\right|^2=\left|\mathrm{Re}\left(\left\langle \boldsymbol{\phi}_2\right|\boldsymbol{W}_{23}\left|\boldsymbol{\phi}_3\right\rangle\right)\right|^2\leqslant\left|\left\langle \boldsymbol{\phi}_2\right|\boldsymbol{W}_{23}\left|\boldsymbol{\phi}_3\right\rangle\right|^2.$$

由柯西-施瓦茨不等式,得

$$\left|\left\langle \boldsymbol{\phi}_2\right|\boldsymbol{W}_{23}\left|\boldsymbol{\phi}_3\right\rangle\right|^2\leqslant\left\langle \boldsymbol{\phi}_2\right|\boldsymbol{W}_{22}\left|\boldsymbol{\phi}_2\right\rangle\left\langle \boldsymbol{\phi}\right|\boldsymbol{W}_{23}^{\dagger}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\left|\boldsymbol{\phi}\right\rangle.$$

当 $\boldsymbol{W}_{22}>0$ 时,由引理1,得

$$\boldsymbol{W}_{33}-\boldsymbol{W}_{23}^{\dagger}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\geqslant 0.$$

所以

$$\left\langle \boldsymbol{\phi}_2\right|\boldsymbol{W}_{22}\left|\boldsymbol{\phi}_2\right\rangle\left\langle \boldsymbol{\phi}\right|\boldsymbol{W}_{23}^{\dagger}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{23}\left|\boldsymbol{\phi}\right\rangle\leqslant\left\langle \boldsymbol{\phi}_2\right|\boldsymbol{W}_{22}\left|\boldsymbol{\phi}_2\right\rangle\left\langle \boldsymbol{\phi}_3\right|\boldsymbol{W}_{33}\left|\boldsymbol{\phi}_3\right\rangle=\left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{|\phi\rangle\langle\phi|}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{|\phi\rangle\langle\phi|}.$$

因此,

$$\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{|\phi\rangle\langle\phi|}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{|\phi\rangle\langle\phi|}\right|^2 \leqslant \left\langle \tilde{\boldsymbol{W}}_{22}\right\rangle_{|\phi\rangle\langle\phi|}\left\langle \tilde{\boldsymbol{W}}_{33}\right\rangle_{|\phi\rangle\langle\phi|}.$$

同理,当$\boldsymbol{W}_{11}>0$,$i=1$,$j=2,3$,有

$$\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{ij}\right\rangle_{|\phi\rangle\langle\phi|}+\left\langle \tilde{\boldsymbol{W}}_{ij}^{\dagger}\right\rangle_{|\phi\rangle\langle\phi|}\right|^2 \leqslant \left\langle \tilde{\boldsymbol{W}}_{ii}\right\rangle_{|\phi\rangle\langle\phi|}\left\langle \tilde{\boldsymbol{W}}_{jj}\right\rangle_{|\phi\rangle\langle\phi|},$$

因此,对任意纯态$|\boldsymbol{\phi}\rangle \in C^3 \otimes C^d$,不等式(29)~(31)成立.

对任意混态$\boldsymbol{\rho}$,$\boldsymbol{\rho}$有谱分解:$\boldsymbol{\rho}=\sum_i p_i|\boldsymbol{\tau}_i\rangle\langle\boldsymbol{\tau}_i|$,$p_i \geqslant 0$,$\sum_i p_i=1$,设

$$\boldsymbol{A}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}+\tilde{\boldsymbol{W}}_{22}\right),\boldsymbol{B}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}-\tilde{\boldsymbol{W}}_{22}\right),\boldsymbol{C}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}+\tilde{\boldsymbol{W}}_{33}\right),$$

$$\boldsymbol{D}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{11}-\tilde{\boldsymbol{W}}_{33}\right),\boldsymbol{E}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{22}+\tilde{\boldsymbol{W}}_{33}\right),\boldsymbol{F}=\frac{1}{2}\left(\tilde{\boldsymbol{W}}_{22}-\tilde{\boldsymbol{W}}_{33}\right).$$

由式(18)和$a_i^2 \geqslant b_i^2+c_i^2$,$\left(\sum_i^n p_i a_i\right)^2 \geqslant \left(\sum_i^n p_i b_i\right)^2+\left(\sum_i^n p_i c_i\right)^2$,其中$b_i$和$c_i$是任意实数,$a_i$是非负的[19],可得

$$\langle \boldsymbol{A}\rangle_{\boldsymbol{\rho}_{\text{sep}}}^2 \geqslant \langle \boldsymbol{B}\rangle_{\boldsymbol{\rho}_{\text{sep}}}^2+\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{12}\right\rangle_{\boldsymbol{\rho}_{\text{sep}}}+\left\langle \tilde{\boldsymbol{W}}_{12}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\text{sep}}}\right|^2, \quad (32)$$

$$\langle \boldsymbol{C}\rangle_{\boldsymbol{\rho}_{\text{sep}}}^2 \geqslant \langle \boldsymbol{D}\rangle_{\boldsymbol{\rho}_{\text{sep}}}^2+\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{13}\right\rangle_{\boldsymbol{\rho}_{\text{sep}}}+\left\langle \tilde{\boldsymbol{W}}_{13}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\text{sep}}}\right|^2, \quad (33)$$

$$\langle \boldsymbol{E}\rangle_{\boldsymbol{\rho}_{\text{sep}}}^2 \geqslant \langle \boldsymbol{F}\rangle_{\boldsymbol{\rho}_{\text{sep}}}^2+\frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{23}\right\rangle_{\boldsymbol{\rho}_{\text{sep}}}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}\right\rangle_{\boldsymbol{\rho}_{\text{sep}}}\right|^2, \quad (34)$$

因式(32)~(34)和式(29)~(31)是等价的,即得式(29)~(31).

可证式(29)~(31)对任意混合态都成立.

若$\boldsymbol{W}_{22} \geqslant 0$,

设$\boldsymbol{W}(\varepsilon)=\boldsymbol{W}+\varepsilon \boldsymbol{I}_{3d}$,$\varepsilon$是任意正数.

由$\boldsymbol{W}(\varepsilon)>0$和讨论$\boldsymbol{W}_{22}>0$情况的方法,有

$$\left\langle \tilde{\boldsymbol{W}}_{22}(\varepsilon)\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{33}(\varepsilon)\right\rangle_{\rho} \geqslant \frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{23}(\varepsilon)\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{23}^{\dagger}(\varepsilon)\right\rangle_{\rho}\right|^2,$$

其中$\tilde{\boldsymbol{W}}_{22}(\varepsilon)$、$\tilde{\boldsymbol{W}}_{33}(\varepsilon)$和$\tilde{\boldsymbol{W}}_{23}(\varepsilon)$定义形如式(7).

当$\varepsilon \to 0$时,式(31)成立.

同理,当$\boldsymbol{W}_{11} \geqslant 0$时,有

$$\left\langle \tilde{\boldsymbol{W}}_{ii}(\varepsilon)\right\rangle_{\rho}\left\langle \tilde{\boldsymbol{W}}_{jj}(\varepsilon)\right\rangle_{\rho} \geqslant \frac{1}{4}\left|\left\langle \tilde{\boldsymbol{W}}_{ij}(\varepsilon)\right\rangle_{\rho}+\left\langle \tilde{\boldsymbol{W}}_{ij}^{\dagger}(\varepsilon)\right\rangle_{\rho}\right|^2,$$

$$i=1,j=2,3.$$

因此,当$\varepsilon \to 0$时,可得式(29)和(30).

例1　若$3 \otimes 3$量子态$\boldsymbol{\rho}_p=p|\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}|+(1-p)\boldsymbol{\rho}$,其中$0 \leqslant p \leqslant 1$,$|\boldsymbol{\varphi}\rangle=\frac{1}{\sqrt{3}}(|00\rangle+|11\rangle+|22\rangle)$,$\boldsymbol{\rho}$是PPT纠缠态,形如[1]:

$$\boldsymbol{\rho}=\frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+a) & 0 & \frac{1}{2}\sqrt{1-a^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{1}{2}\sqrt{1-a^2} & 0 & \frac{1}{2}(1+a) \end{pmatrix},$$

其中$0 \leqslant a \leqslant 1$.

纠缠见证$\boldsymbol{W}=\alpha \boldsymbol{I}_{3\times 3}-|\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}|$,$\alpha$是$|\boldsymbol{\varphi}\rangle$的最大施密特系数的平方.$\boldsymbol{W}$经过式(25)~(27)非线性改进后对$\boldsymbol{\rho}_p$的纠缠判别如表1所示.

表1　对a的不同取值下的$\boldsymbol{\rho}_p$的纠缠判别

态的参数(a)	纠缠范围(p)
0.1	[0.071 8,1.000 0]
0.2	[0.096 7,1.000 0]
0.3	[0.109 4,1.000 0]
0.4	[0.117 1,1.000 0]

例2　若$3 \otimes 3$量子态$\boldsymbol{\rho}_p=p|\boldsymbol{\varphi}\rangle\langle\boldsymbol{\varphi}|+(1-p)\boldsymbol{I}_9$,其中$0 \leqslant p \leqslant 1$,$|\boldsymbol{\varphi}\rangle=\frac{1}{\sqrt{3}}(|00\rangle+|01\rangle+|10\rangle)$.纠缠见证$\boldsymbol{W}=\sum_{i,j=0}^{2}|i\rangle\langle j|\otimes|j\rangle\langle i|$.经计算,$\langle \boldsymbol{W}\rangle_{\boldsymbol{\rho}_p}>0$,根据定义4,$\boldsymbol{W}$不能判别$\boldsymbol{\rho}_p$的任意纠缠.$\boldsymbol{W}$经过式(25)~(27)非线性改进后,可以判别$0.548\,2 \leqslant p \leqslant 1.000\,0$范围内$\boldsymbol{\rho}_p$的纠缠.

3　结　论

本文研究了$C^3 \otimes C^d$量子系统中的纠缠见证,给出了其非线性改进,不仅能用较少的观测量判别纠缠,而且改进后的纠缠见证能判别出更多的纠缠态,并用实例验证了此方法有较好的优越性.

(1)给出了厄米算子作为纠缠见证的充要条件;

(2)研究了$C^3 \otimes C^d$上算子半正定的充要条件,并证明了对满足条件$\langle \boldsymbol{W}\rangle_{\rho}<0$的任意态$\boldsymbol{\rho}$,可用非线性改进后的纠缠见证进行判别;

(3)给出对$C^3 \otimes C^d$中任意态,$\boldsymbol{W}$半正定的等价条件,并用例子说明经过非线性改进后的纠缠见证

可以判别量子态是否纠缠,且能判别出更多的纠缠态.

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