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一类带有积分边界条件和变号非线性项的四阶 p-Laplacian 微分方程解的存在唯一性*

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摘 要

利用上下解方法和 Leray-Schauder 度理论,研究了四阶 p-Laplacian 微分方程$(\phi(u'''(t)))'-f(t,u(t),u'(t),u''(t),u'''(t))=0, t\in(0,1)$ 在积分边界条件下解的存在性和唯一性. 其中 $f:[0,1]\times R^4\to R$ 为连续函数,$\phi(u)$ 为增同胚且 $\phi(0)=0,\phi(R)=R,R=(-\infty,+\infty)$.

关键词: 积分边界条件,上下解,Leray-Schauder 度理论,Nagumo 条件,p-Laplacian.
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1 引 言

众所周知,弹性梁的弯曲问题可以被描述为一些四阶常微分方程边值问题. 基于弹性梁的弯曲问题在物理学中的重要性,人们对它的研究十分活跃,例如[1-4]. 文[4]给出了四阶微分方程

$$\begin{cases} u'''(t)-f(t,u(t),u'(t),u''(t),u'''(t))=0, \\ \qquad\qquad 0<t<1 \\ u(0)=u'(1)=au''(0)-bu'''(0) \\ \qquad\quad =cu''(1)+du'''(1)=0, \end{cases}$$

解的存在性和唯一性. 其中 $a,b,c,d\geqslant 0,\rho: ad+bc+ac>0$ 且 $f:[0,1]\times R^4\to R$ 为连续函数.

与此同时,我们注意到带有积分边界条件的边值问题出现在热传导,化学工程,地下水流,热力弹性和等离子体物理中. 这些问题包括两个,三个,多点和非局部的边值问题作为特殊情况. 其广泛的应用背景引起了许多作者对此类方程解的存在性的关注,见文献[5-6]. 鉴于带有积分边界条件的边值问题的重要性,受文[4]启发本文讨论带有积分边界条件和 p-Laplacian 的四阶边值问题

$$\begin{cases} (\phi(u'''(t)))'-f(t,u(t),u'(t),u''(t), \\ \quad u'''(t))=0,\ 0<t<1, \\ u(0)=u'(1)=0, \\ u''(0)-k_1u'''(0)=\int_0^1 h_1(u(s),u'(s),u''(s))ds, \\ u''(1)+k_2u'''(1)=\int_0^1 h_2(u(s),u'(s),u''(s))ds, \end{cases} \tag{1.1}$$

$$\begin{cases} (\phi(u'''(t)))'-f(t,u(t),u'(t),u''(t), \\ \quad u'''(t))=0,\ 0<t<1, \\ u(0)=u'(1)=0, \\ u''(0)-k_1u'''(0)=\int_0^1 g_1(u''(s))ds, \\ u''(1)+k_2u'''(1)=\int_0^1 g_2(u''(s))ds, \end{cases} \tag{1.2}$$

其中 $f:[0,1]\times R^4\to R$ 为连续函数;ϕ 为增同胚且 $\phi(0)=0,\phi(R)=R$;$h_i:R^3\to R, g_i:R\to R(i=1,2)$ 为连续函数;$k_i\geqslant 0(i=1,2)$;$R=(-\infty,+\infty)$.

本文与文[5]比较有如下不同点. 第一,本文利用上下解方法和 Leray-Schauder 度理论,研究了方程(1.1)解的存在性以及方程(1.2)解的存在性和唯一性,这是不同于文[5]的. 第二,本文讨论的是带有积分边界条件的边值问题,这要比[5]讨论的边值问题更广泛. 第三,本文带有 p-Laplacian 算子而文[5]没有.

2 预备知识及引理

定义1 若函数 $\alpha(t)\in C^3[0,1]$,$\phi(\alpha'''(t))\in$

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$C^1[0,1]$ 满足

$$(\phi(\alpha'''(t)))' \geqslant f(t,\alpha(t),\alpha'(t),\alpha''(t),\alpha'''(t)) \leqslant 0,\quad 0<t<1, \tag{2.1}$$

且满足

$$\begin{cases}\alpha(0)\leqslant 0,\ \alpha'(1)\leqslant 0,\\ \alpha''(0)-k_1\alpha'''(0)\geqslant\int_0^1 h_1(\alpha(s),\alpha'(s),\alpha''(s))\,ds,\\ \alpha''(1)+k_2\alpha'''(1)\geqslant\int_0^1 h_2(\alpha(s),\alpha'(s),\alpha''(s))\,ds,\end{cases} \tag{2.2}$$

则称 $\alpha(t)$ 为边值问题(1.1) 的一个下解; 上解 $\beta(t)$ 的定义只要将上述不等式反向.

定义 2　若函数 $m(t)\in C^3[0,1]$, $\phi(m'''(t))\in C^1[0,1]$ 满足

$$(\phi(m'''(t)))' \geqslant f(t,m(t),m'(t),m''(t),m'''(t)) \leqslant 0,\quad 0<t<1, \tag{2.3}$$

且满足

$$\begin{cases}m(0)\leqslant 0,\ m'(1)\leqslant 0,\\ m''(0)-k_1m'''(0)\geqslant\int_0^1 g_1(m''(s))\,ds,\\ m''(1)+k_2m'''(1)\geqslant\int_0^1 g_2(m''(s))\,ds,\end{cases} \tag{2.4}$$

则称 $m(t)$ 为边值问题(1.2) 的一个下解; 上解 $n(t)$ 的定义只要将上述不等式反向.

定义 3　令

$$D=\{(t,w,x,y,z)\in[0,1]\times R^4: a(t)\leqslant w\leqslant b(t),c(t)\leqslant x\leqslant d(t),\gamma(t)\leqslant y\leqslant\delta(t)\},$$

如果对于连续函数 $f:[0,1]\times R^4\to R$ 存在一个连续函数 $\Phi_D:[0,\infty]\to[a,\infty]$, $a>0$ 以及常数 $p>1$ 使得对于 $\forall(t,w,x,y,z)\in D$ 满足

$$|f(t,w,x,y,z)|\leqslant\Phi_D(|z|), \tag{2.5}$$

而且

$$\int_{-\infty}^{\phi(-v)}\frac{|\phi^{-1}(s)|^{(p-1)/p}}{\Phi_D(|\phi^{-1}(s)|)}ds,\quad \int_{\phi(v)}^{+\infty}\frac{|\phi^{-1}(s)|^{(p-1)/p}}{\Phi_D(|\phi^{-1}(s)|)}ds=+\infty, \tag{2.6}$$

其中 ϕ^{-1} 为函数 ϕ 的反函数, $v=\max\{|\delta(1)-\gamma(0)|,|\gamma(1)-\delta(0)|\}$, 则称连续函数 f 在 D 上满足 Nagumo 条件.

引理 1　若连续函数 $f:[0,1]\times R^4\to R$ 在

$$D=\{(t,w,x,y,z)\in[0,1]\times R^4: a(t)\leqslant w\leqslant b(t),c(t)\leqslant x\leqslant d(t),\gamma(t)\leqslant y\leqslant\delta(t)\}$$

上满足 Nagumo 条件, 其中 $a,b,c,d,\gamma,\delta:[0,1]\to R$ 为连续函数且 $a(t)\leqslant b(t)$, $c(t)\leqslant d(t)$, $\gamma(t)\leqslant\delta(t)$, $t\in[0,1]$, 则存在 $N>0$(与 γ,δ,Φ_D 有关), 使得(1.1) 或(1.2) 所得满足条件

$$a(t)\leqslant u(t)\leqslant b(t),c(t)\leqslant u'(t)\leqslant d(t),\gamma(t)\leqslant u''(t)\leqslant\delta(t),t\in[0,1]$$

的解 $u(t)$ 都有 $\|u'''\|_\infty<N$.

证明: 根据(2.6) 可取常数 N 足够大, 以至于 $N>v$, 而且

$$\int_{-\infty}^{\phi(-v)}\frac{|\phi^{-1}(s)|^{(p-1)/p}}{\Phi_D(|\phi^{-1}(s)|)}ds,\quad \int_{\phi(v)}^{+\infty}\frac{|\phi^{-1}(s)|^{(p-1)/p}}{\Phi_D(|\phi^{-1}(s)|)}ds>\mu^{(p-1)/p}, \tag{2.7}$$

其中 $\mu=\max_{t\in[0,1]}\delta(t)-\max_{t\in[0,1]}\gamma(t)$.

根据中值定理, 存在 $t_0\in(0,1)$ 使得 $u'''(t_0)=u''(1)-u''(0)$. 很显然如下式子成立: $-N<-v<\gamma(1)-\delta(0)\leqslant u'''(t_0)\leqslant\delta(1)-\gamma(0)\leqslant v<N$.

定义　$v_0=|u'''(t_0)|$. 我们断言 $\|u'''\|_\infty<N$. 如果反之, 根据 $u'''(t)$ 的连续性则存在 $[t_1,t_2]\in[0,1]$ 有下述四种可能

(i) $u'''(t_1)=v_0$, $u'''(t_2)=N$, $v_0\leqslant u'''(t)\leqslant N$, $t\in(t_1,t_2)$;

(ii) $u'''(t_1)=N$, $u'''(t_2)=v_0$, $v_0\leqslant u'''(t)\leqslant N$, $t\in(t_1,t_2)$;

(iii) $u'''(t_1)=-v_0$, $u'''(t_2)=-N$, $-N\leqslant u'''(t)\leqslant -v_0$, $t\in(t_1,t_2)$;

(iv) $u'''(t_1)=-N$, $u'''(t_2)=-v_0$, $-N\leqslant u'''(t)\leqslant -v_0$, $t\in(t_1,t_2)$.

假设(i) 满足. 根据(2.5) 我们有

$$|(\phi(u'''(t)))'|=|f(t,u(t),u'(t),u''(t),u'''(t))|\leqslant\Phi_D(|u'''(t)|),\ t\in(t_1,t_2).$$

根据上式我们有

$$\int_{\phi(v_0)}^{\phi(N)}\frac{|\phi^{-1}(s)|^{(p-1)/p}}{\Phi_D(|\phi^{-1}(s)|)}ds=\int_{t_1}^{t_2}\frac{|u'''(t)|^{(p-1)/p}}{\Phi_D(|u'''(t)|)}(\phi(u'''(t)))'dt\leqslant\int_{t_1}^{t_2}|u'''(t)|^{(p-1)/p}dt\leqslant(t_2-t_1)^{1/p}(u''(t_2)-u''(t_1))^{(p-1)/p}\leqslant\mu^{(p-1)/p},$$

这与(2.7) 矛盾. 同理可证(ii),(iii),(iv) 与(2.7) 矛盾. 所以 $\|u'''\|_\infty<N$. 证毕.

引理 2　我们考虑如下方程

$$\begin{cases}(\phi(u'''(t)))'=u''(t),\quad 0<t<1,\\ u(0)=u'(1)=u''(0)=u''(1)=0,\end{cases} \tag{2.8}$$

则方程(2.8) 只有平凡解.

证明: 令 $u''(t)=v(t)$, 则方程(2.8) 可化为:

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$$\begin{cases}(\phi(v'(t)))' = v(t), & 0 < t < 1,\\ v(0) = v(1) = 0.\end{cases} \tag{2.9}$$

下面我们证方程(2.9)只有平凡解.

显然 $v(t) \equiv 0$ 为方程(2.9)的解. 假设 $v(t)$ 为方程(2.9)的非平凡解,则存在 $t^* \in (0,1)$ 使得 $v(t^*) \neq 0$. 不失一般性,我们假设 $v(t^*) > 0$,则存在 $[t_1,t_2] \in [0,1]$ 使得

$$v(t) > 0, t \in (t_1,t_2), v(t_1) = v(t_2) = 0.$$

所以存在 $t_0 \in (t_1,t_2)$,使得 $v(t_0) = \max\limits_{t\in[t_1,t_2]} v(t) > 0$. 则 $v'(t_0) = 0$ 且存在 $\delta > 0$ 使得 $(t_0-\delta,t_0+\delta) \subset (t_1,t_2)$,

$$\begin{aligned} &v(t) > 0, t \in (t_0-\delta,t_0+\delta),\\ &v'(t_3) \geqslant 0, t_3 \in (t_0-\delta,t_0),\\ &v'(t_4) \leqslant 0, t_4 \in (t_0,t_0+\delta).\end{aligned}$$

利用 $t_3,t_4 \in (t_0-\delta,t_0+\delta)$, $t_3 < t_4$,(2.9)可得 $\phi(v'(t_3)) < \phi(v'(t_4))$,则 $v'(t_3) < v'(t_4)$,矛盾. 所以方程(2.9)只有平凡解.

根据 $u''(t) = v(t)$ 以及 $u(0) = u'(1) = 0$ 可得方程(2.8)只有平凡解.

3 主要结果

(H_1) 方程(1.1)存在上下解 $\beta(t)$, $\alpha(t)$ 而且 $\beta''(t) \leqslant \alpha''(t)$, $t \in [0,1]$;

(H_2) $f \in C([0,1] \times R^4, R)$ 在 E 上满足 Nagumo 条件,其中

$$E = \{(t,w,x,y,z) \in [0,1] \times R^4 : \alpha(t) \leqslant w \leqslant \beta(t), \alpha'(t) \leqslant x \leqslant \beta'(t), \beta''(t) \leqslant y \leqslant \alpha''(t)\};$$

(H_3) $h_i \in C(R^3,R)$ $(i = 1,2)$ 关于第一与第二个变元非增,关于第三个变元非减;

(H_4) 对于 $(t,y,z) \in [0,1] \times R^2$, $\alpha(t) \leqslant w \leqslant \beta(t)$, $\alpha'(t) \leqslant x \leqslant \beta'(t)$ 有下列成立

$$f(t,\alpha(t),\alpha'(t),y,z) \leqslant f(t,w,x,y,z) \leqslant f(t,\beta(t),\beta'(t),y,z);$$

(H_5) $\phi(u)$ 为增同胚且 $\phi(0) = 0$, $\phi(R) = R$, $k_i \geqslant 0$ $(i = 1,2)$.

定理 3.1 若条件 (H_1)—(H_3) 成立,则边值问题(1.1)至少有一个解 $u(t)$ 满足

$$\alpha(t) \leqslant u(t) \leqslant \beta(t), \alpha'(t) \leqslant u'(t) \leqslant \beta'(t), \beta''(t) \leqslant u''(t) \leqslant \alpha''(t), |u'''(t)| \leqslant N, t \in [0,1],$$

其中 N 为 α,β,Φ_E 有关的常数.

证明: 对于 $i = 0,1$, $t \in [0,1]$ 定义

$$\delta_i(t,x_i) = \begin{cases}\beta^{(i)}(t), & x_i > \beta^{(i)}(t),\\ x_i, & \alpha^{(i)}(t) \leqslant x_i \leqslant \beta^{(i)}(t),\\ \alpha^{(i)}(t), & x_i < \alpha^{(i)}(t),\end{cases}$$

$$\delta_2(t,x_2) = \begin{cases}\alpha''(t), & x_2 > \alpha''(t),\\ x_2, & \beta''(t) \leqslant x_2 \leqslant \alpha''(t),\\ \beta''(t), & x_2 < \beta''(t).\end{cases}$$

由 (H_2) 知存在一个连续函数 $\Phi_E: [0,\infty] \to [b,\infty]$, $b > 0$ 以及常数 $p_1 > 1$ 使得对于 $\forall (t,w,x,y,z) \in E$ 满足

$$|f(t,w,x,y,z)| \leqslant \Phi_E(|z|), \tag{3.1}$$

并且

$$\begin{aligned}&\int_{-\infty}^{\phi(-\theta)} \frac{|\phi^{-1}(s)|^{(p_1-1)/p_1}}{\Phi_E(|\phi^{-1}(s)|)}ds,\\ &\int_{\phi(\theta)}^{+\infty} \frac{|\phi^{-1}(s)|^{(p_1-1)/p_1}}{\Phi_E(|\phi^{-1}(s)|)}ds = +\infty.\end{aligned} \tag{3.2}$$

其中 ϕ^{-1} 为函数 ϕ 的反函数,$\theta = \max\{|\beta''(1) - \alpha''(0)|, |\alpha''(1) - \beta''(0)|\}$,

对于 $\lambda \in [0,1]$, $t \in [0,1]$ 我们考虑如下方程:

$$\begin{aligned}(\phi(u'''(t)))' = &\lambda f(t,\delta_0(t,u(t)),\delta_1(t,u'(t)),\\ &\delta_2(t,u''(t)),u'''(t))) + (1-\lambda)u''(t) +\\ &\lambda[u''(t) - \delta_2(t,u''(t))]\Phi_E(|u'''(t)|),\end{aligned} \tag{3.3}$$

方程(3.3)带有如下边值条件

$$\begin{cases}u(0) = u'(1) = 0,\\ u''(0) = \lambda\left[\int_0^1 h_1(\delta_0(s,u(s)),\delta_1(s,u'(s)),\right.\\ \qquad \left.\delta_2(s,u''(s)))ds + k_1u'''(0)\right],\\ u''(1) = \lambda\left[\int_0^1 h_2(\delta_0(s,u(s)),\delta_1(s,u'(s)),\right.\\ \qquad \left.\delta_2(s,u''(s)))ds - k_2u'''(0)\right].\end{cases} \tag{3.4}$$

取 $M_1 > 0$ 使得下列式子在 $t \in [0,1]$ 上成立:

$$-M_1 < \beta''(t) \leqslant \alpha''(t) < M_1, \tag{3.5}$$

$$f(t,\alpha(t),\alpha'(t),\alpha''(t),0) + [M_1 - \alpha''(t)]\Phi_E(0) > 0, \tag{3.6}$$

$$f(t,\beta(t),\beta'(t),\beta''(t),0) - [M_1 + \beta''(t)]\Phi_E(0) < 0, \tag{3.7}$$

$$\begin{aligned}&\left|\int_0^1 h_1(\alpha(s),\alpha'(s),\alpha''(s))ds\right| < M_1,\\ &\left|\int_0^1 h_1(\beta(s),\beta'(s),\beta''(s))ds\right| < M_1,\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\left|\int_0^1 h_2(\alpha(s),\alpha'(s),\alpha''(s))ds\right| < M_1,\\ &\left|\int_0^1 h_2(\beta(s),\beta'(s),\beta''(s))ds\right| < M_1,\end{aligned} \tag{3.9}$$

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第一步,对于 $\forall \lambda \in [0,1]$,(3.3),(3.4) **的解** $u(t)$ **必满足:**

$|u(t)| < M_1, |u'(t)| < M_1, |u''(t)| < M_1, t \in [0,1]$.

如果 $|u''(t)| < M_1$ **不成立,则存在** $t \in [0,1]$ **使得** $u''(t) \leqslant -M_1$ **或** $u''(t) \geqslant M_1$. **假设后者成立,定义**

$$\max_{t\in[0,1]} u''(t) = u''(t_0) (\geqslant M_1 > 0). \quad (3.10)$$

(1) **如果** $t_0 \in (0,1)$,**则** $u'''(t_0) = 0$. **由**(3.3),(3.10),(H_4) **得**

$$\begin{aligned}
0 &= (\phi(u'''(t_0)))' \\
&= \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \\
&\quad \delta_2(t_0, u''(t_0)), u'''(t_0)) + (1-\lambda)u''(t_0) + \\
&\quad \lambda [u''(t_0) - \delta_2(t_0, u''(t_0))] \Phi_E(|u'''(t_0)|) \\
&\geqslant \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \alpha''(t_0), 0) \\
&\quad + (1-\lambda)M_1 + \lambda [M_1 - \alpha''(t_0)] \Phi_E(0) \\
&\geqslant \lambda f(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0), 0) + (1-\lambda)M_1 + \\
&\quad \lambda [M_1 - \alpha''(t_0)] \Phi_E(0) \\
&= \lambda [f(t_0, \alpha(t_0), \alpha'(t_0), \alpha''(t_0), 0) + \\
&\quad [M_1 - \alpha''(t_0)] \Phi_E(0)] + (1-\lambda)M_1 > 0,
\end{aligned}$$

矛盾. 所以 t_0 **不属于**$(0,1)$.

(2) **如果** $t_0 = 0$,**则** $\max_{t\in[0,1]} u''(t) = u''(0) (\geqslant M_1 > 0)$,$u'''(0^+) = u'''(0) \leqslant 0$. **由**(3.4),(3.8),$(H_3)$ **可得**

$$\begin{aligned}
M_1 \leqslant u''(0) &= \lambda \left[\int_0^1 h_1(\delta_0(s,u(s)), \delta_1(s,u'(s)), \right. \\
&\quad \left. \delta_2(s,u''(s)))\,ds + k_1 u'''(0)\right] \\
&\leqslant \int_0^1 h_1(\delta_0(s,u(s)), \delta_1(s,u'(s)), \\
&\quad \delta_2(s,u''(s)))\,ds + k_1 u'''(0) \\
&\leqslant \int_0^1 h_1(\alpha(s), \alpha'(s), \alpha''(s))\,ds < M_1,
\end{aligned}$$

矛盾.

(3) **如果** $t_0 = 1$,**则** $\max_{t\in[0,1]} u''(t) = u''(1) (\geqslant M_1 > 0)$,$u'''(0^-) = u'''(1) \geqslant 0$. **由**(3.4),(3.9),$(H_3)$ **可得**

$$\begin{aligned}
M_1 \leqslant u''(1) &= \lambda \left[\int_0^1 h_2(\delta_0(s,u(s)), \delta_1(s,u'(s)), \right. \\
&\quad \left. \delta_2(s,u''(s)))\,ds - k_2 u'''(0)\right] \\
&\leqslant \int_0^1 h_2(\delta_0(s,u(s)), \delta_1(s,u'(s)), \\
&\quad \delta_2(s,u''(s)))\,ds - k_2 u'''(1) \\
&\leqslant \int_0^1 h_2(\alpha(s), \alpha'(s), \alpha''(s))\,ds < M_1,
\end{aligned}$$

矛盾. 假设不成立.

用同样方法可证存在 $t \in [0,1]$ **使得** $u''(t) \leqslant -M_1$ **这个假设也不成立. 所以** $|u''(t)| < M_1 (t \in [0,1])$ **成立. 通过积分以及** $u(0) = u'(1) = 0$ **可得** $|u(t)| < M_1, |u'(t)| < M_1$.

第二步,存在 $M_2 > 0$,**使得对于** $\forall \lambda \in [0,1]$,(3.3)—(3.4) **的解必满足:** $|u'''(t)| < M_2, t \in [0,1]$.

如果 $u(t)$ **为**(3.3),(3.4) **的解,则**

$$\begin{aligned}
(\phi(u'''(t)))' &= \lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \\
&\quad \delta_2(t, u''(t)), u'''(t)) + \\
&\quad (1-\lambda)u''(t) + \lambda [u''(t) - \\
&\quad \delta_2(t, u''(t))] \Phi_E(|u'''(t)|).
\end{aligned}$$

令

$$\begin{aligned}
S = \{&(t,w,x,y,z) \in [0,1] \times R^4 : -M_1 \leqslant w \leqslant M_1, \\
&-M_1 < x < M_1, -M_1 \leqslant y \leqslant M_1\}.
\end{aligned}$$

定义 $F_\lambda : S \to R$ **且**

$$\begin{aligned}
F_\lambda(t,w,x,y,z) &= \lambda f(t, \delta_0(t,w), \delta_1(t,x), \delta_2(t,y,z) \\
&\quad + (1-\lambda)y + \lambda [y - \delta_2(t,y)] \Phi_E(|z|).
\end{aligned}$$

下面我们证明对于 $\forall \lambda \in [0,1]$ $F_\lambda(t,w,x,y,z)$ **在** S **上满足** Nagumo **条件. 由于** f **在** E **上满足** Nagumo **条件,则由**(3.1) **有**

$$\begin{aligned}
|F_\lambda(t,w,x,y,z)| &\leqslant |f(t, \delta_0(t,w), \delta_1(t,x), \\
&\quad \delta_2(t,y), z)| + |y| + |y - \\
&\quad \delta_2(t,y)| \Phi_E(|z|) \\
&\leqslant \Phi_E(|z|) + M_1 + (|y| + \\
&\quad |\delta_2(t,y)|) \Phi_E(|z|) \\
&\leqslant M_1 + (1 + 2M_1) \Phi_E(|z|) \\
&= \Phi_S(|z|). \qquad (3.11)
\end{aligned}$$

由上式以及(3.2) **我们得**

$$\begin{aligned}
&\int_{\phi(\theta)}^{+\infty} \frac{|\phi^{-1}(s)|^{(p_1-1)/p_1}}{\Phi_s(|\phi^{-1}(s)|)} ds \\
&= \int_{\phi(\theta)}^{+\infty} \frac{|\phi^{-1}(s)|^{(p_1-1)/p_1}}{M_1 + (1+2M_1)\Phi_E(|\phi^{-1}(s)|)} ds \\
&\geqslant \int_{\phi(\theta)}^{+\infty} \frac{|\phi^{-1}(s)|^{(p_1-1)/p_1}}{\left(1 + 2M_1 + \frac{M_1}{b}\right)\Phi_E(|\phi^{-1}(s)|)} ds \\
&= \frac{1}{1 + 2M_1 + \frac{M_1}{b}} \int_{\phi(\theta)}^{+\infty} \frac{|\phi^{-1}(s)|^{(p_1-1)/p_1}}{\Phi_E(|\phi^{-1}(s)|)} ds \\
&= +\infty.
\end{aligned}$$

用同样方法可证

$$\int_{-\infty}^{\phi(-\theta)} \frac{|\phi^{-1}(s)|^{(p_1-1)/p_1}}{\Phi_S(|\phi^{-1}(s)|)} ds = +\infty.$$

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由(3.11),(3.12),(3.13)可知对于 $\forall \lambda \in [0,1]$,$F_\lambda(t,w,x,y,z)$ 在 S 上满足 Nagumo 条件. 由第一步的结论与引理 1 我们可得存在 $M_2 > 0$,使得对于 $\forall \in [0,1]$,(3.3)—(3.4) 的解 $u(t)$ 必满足: $|u'''(t)| < M_2, t \in [0,1]$.

第三步,对于 $\lambda = 1$ 方程(3.3)—(3.4) 至少有一个解 $u_1(t)$.

定义算子 $M: C^3[0,1] \cap \mathrm{dom}M \to C[0,1] \times R^4$,并且

$$Mu = ((\phi(u'''(t)))', u(0), u'(1), u''(0), u''(1)),$$

对于 $\lambda \in [0,1]$ 定义 $N_\lambda: C^3[0,1] \to C[0,1] \times R^4$,并且

$$\begin{aligned} N_\lambda(u) &= \lambda f(t,\delta_0(t,u(t)),\delta_1(t,u'(t)),\\ &\quad \delta_2(t,u''(t)),u'''(t)) + (1-\lambda)u'(t)\\ &\quad + \lambda[u''(t) - \delta_2(t,u''(t))] \times\\ &\quad \Phi_E(|u'''(t)|),0,0,A_\lambda,B_\lambda), \end{aligned}$$

其中

$$\begin{aligned} A_\lambda &= \lambda\left[\int_0^1 h_1(\delta_0(s,u(s)),\delta_1(s,u'(s)),\right.\\ &\quad \left.\delta_2(s,u''(s)))ds + k_1u'''(0)\right],\\ B_\lambda &= \lambda\left[\int_0^1 h_2(\delta_0(s,u(s)),\delta_1(s,u'(s)),\right.\\ &\quad \left.\delta_2(s,u''(s)))ds - k_2u'''(1)\right]. \end{aligned}$$

因为 $\phi(u)$ 为增同胚且 $\phi(0) = 0, \phi(R) = R$,所以 M 有反函数且 M^{-1} 为紧算子. 又由于 N_λ 连续,所以我们定义全连续算子 $T_\lambda: C^3[0,1] \to C^3[0,1]$ 并且 $T_\lambda(u) = M^{-1}N_\lambda(u)$.

考虑

$$\Omega = \{u \in C^3[0,1]: \|u^{(i)}\|_\infty < M_1, i = 0,1,2, \|u'''\|_\infty < M_2\}.$$

由第一步与第二步结论,对于 $\forall \lambda \in [0,1]$,$d(I - T_\lambda,\Omega,0)$ 良定义,根据 Leray-Schauder 同伦不变性得 $d(I - T_0,\Omega,0) = d(I - T_1,\Omega,0)$. 方程 $u = T_0(u)$ 等价于

$$\begin{cases} (\phi(u'''(t)))' = u''(t), 0 < t < 1,\\ u(0) = u'(1) = u''(0) = u''(1) = 0. \end{cases}$$

由引理 2 知上述方程只有平凡解,再由 Lerary-Schauder 度理论得 $|d(I - T_0,\Omega,0)| = 1$,所以 $|d(I - T_1,\Omega,0)| = 1$,所以 $u = T_1(u)$ 在 Ω 中至少有一个解. 也就是对于 $\lambda = 1$ 方程(3.3)—(3.4) 在 Ω 中至少有一个解 $u_1(t)$.

第四步,$u_1(t)$ 为方程(1.1) 的解.

要证 $u_1(t)$ 为方程(1.1) 的解,只需证

$\alpha(t) \leqslant u_1(t) \leqslant \beta(t), \alpha'(t) \leqslant u_1'(t) \leqslant \beta'(t)$, $\beta''(t) \leqslant u_1''(t) \leqslant \alpha''(t), t \in [0,1]$.

假设存在 $t \in [0,1]$ 使得 $u_1''(t) > \alpha''(t)$ 那以定义 $u_1''(s_0) - \alpha''(s_0) = \max\limits_{t \in [0,1]}[u_1''(t) - \alpha''(t)] > 0$.

(1) 如果 $s_0 \in (0,1)$,则 $u_1'''(s_0) - \alpha'''(s_0) = 0$,则由(2.1),(3.3),($H_4$) 得

$$\begin{aligned} (\phi(u_1'''(s_0)))' &= f(s_0,\delta_0(s_0,u(s_0)),\delta_1(s_0,u'(s_0)),\\ &\quad \delta_2(s_0,u''(s_0)),u'''(s_0)) +\\ &\quad [u''(s_0) - \delta_2(s_0,u''(s_0))] \times\\ &\quad \Phi_E(|u''(s_0)|)\\ &= f(s_0,\delta_0(s_0,u(s_0)),\delta_1(s_0,u'(s_0)),\\ &\quad \alpha''(s_0),\alpha'''(s_0)) + [u''(s_0) - \alpha''(s_0)],\\ &\quad \Phi_E(|u'''(s_0)|)\\ &> f(s_0,\alpha(s_0),\alpha'(s_0),\alpha''(s_0),\\ &\quad \alpha'''(s_0)) \geqslant (\phi(\alpha'''(s_0)))', \text{矛盾}. \end{aligned}$$

(2) 如果 $s_0 = 0$,则 $u_1''(0) - \alpha''(0) = \max\limits_{t \in [0,1]}[u_1''(t) - \alpha''(t)] > 0, u_1'''(0^+) - \alpha'''(0^+) = u_1'''(0) - \alpha'''(0) \leqslant 0$,则由(2.2),(3.4),($H_3$) 得

$$\begin{aligned} \alpha''(0) < u_1''(0) &= \int_0^1 h_1(\delta_0(s,u_1(s)),\delta_1(s,u_1'(s)),\\ &\quad \delta_2(s,u_1''(s)))ds + k_1u_1'''(0)\\ &\leqslant \int_0^1 h_1(\alpha(s),\alpha'(s),\alpha''(s)ds +\\ &\quad k_1\alpha'''(0) \leqslant \alpha''(0), \text{矛盾}. \end{aligned}$$

(3) 如果 $s_0 = 1$,则 $u_1''(1) - \alpha''(1) = \max\limits_{t \in [0,1]}[u_1''(t) - \alpha''(t)] > 0, u_1'''(1^-) - \alpha'''(1^-) = u_1'''(1) - \alpha'''(1) \geqslant 0$,则由(2.2),(3.4),($H_3$) 得

$$\begin{aligned} \alpha''(1) < u_1''(1) &= \int_0^1 h_1(\delta_0(s,u_1(s)),\delta_1(s,u_1'(s)),\\ &\quad \delta_2(s,u_1''(s)))ds - k_2u_1'''(1)\\ &\leqslant \int_0^1 h_2(\alpha(s),\alpha'(s),\alpha''(s)ds -\\ &\quad k_2\alpha'''(1) \leqslant \alpha''(1). \text{矛盾}. \end{aligned}$$

所以 $u_1''(t) \leqslant \alpha''(t)$ ($t \in [0,1]$). 用同样方法可证 $\beta''(t) \leqslant u_1''(t)$ ($t \in [0,1]$). 通过积分以及 $u_1(0) = u_1'(1) = 0$ 可得 $\alpha(t) \leqslant u_1(t) \leqslant \beta(t)$, $\alpha'(t) \leqslant u_1'(t) \leqslant \beta'(t)$ ($t \in [0,1]$). 所以 $u_1(t)$ 为方程(1.1) 的解. 证毕.

(H_6) 方程(1.2) 存在上下解 $n(t), m(t)$;

(H_7) $f \in C([0,1] \times R^4, R)$ 在 H 上满足 Nagumo 条件,其中

$$H = \{(t,w,x,y,z) \in [0,1] \times R^4: m(t) \leqslant w \leqslant n(t), m'(t) \leqslant x \leqslant n'(t), n''(t) \leqslant y \leqslant m''(t)\};$$

(H_8) $f_y(t,w,x,y,z) \geqslant l$ ($l > 0$ 为常数),$g_i \in C(R,R)$ ($i = 1,2$), $0 \leqslant g'_i(x) < 1$;

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(H_9) 对于$(t,y,z) \in [0,1] \times R^2, m(t) \leqslant w \leqslant n(t), m'(t) \leqslant x \leqslant n'(t)$ 有下式成立

$$f(t,m(t),m'(t),y,z) \leqslant f(t,w,x,y,z) \leqslant f(t,n(t),n'(t),y,z);$$

(H_{10}) $\phi(u)$ 为增同胚且 $\phi(0)=0, \phi(R)=R, k_i \geqslant 0 (i=1,2)$.

定理 3.2　若条件 (H_6)—(H_{10}) 成立,则边值问题(1.2) 有唯一解 $u(t)$ 满足

$$m(t) \leqslant u(t) \leqslant n(t), m'(t) \leqslant u'(t) \leqslant n'(t), n''(t) \leqslant u''(t) \leqslant m''(t), |u'''(t)| \leqslant N, t \in [0,1],$$

其中 N 为与 m,n,Φ_H 有关的常数.

证明: 第一步,方程(1.2) 的上下解 $n(t), m(t)$ 满足 $n''(t) \leqslant m''(t)$. 令 $r(t)=n''(t)-m''(t)$,由于(2.3)(2.4) 我们有

$$r(0)-k_1 r'(0) \leqslant \int_0^1 [g_1(n''(s))-g_1(m''(s))]\,ds,$$

$$r(1)+k_2 r'(1) \leqslant \int_0^1 [g_2(n''(s))-g_2(m''(s))]\,ds. \tag{3.14}$$

如果 $r(t) \leqslant 0, t \in [0,1]$ 不成立. 则设 $\max\limits_{t\in[0,1]}[n''(t)-m''(t)] = n''(c_0)-m''(c_0) > 0$.

(1) 如果 $c_0 \in (0,1)$ 则 $n'''(c_0)-m'''(c_0)=0$,则由(2.3),(H_8),(H_9) 得

$$\begin{aligned}0 &= (\phi(n'''(c_0)))' - (\phi(m'''(c_0)))' \\ &\geqslant f(c_0,n(c_0),n'(c_0),n''(c_0),n'''(c_0)) - f(c_0,m(c_0),m'(c_0),m''(c_0),m'''(c_0)) \\ &= f(c_0,n(c_0),n'(c_0),n''(c_0),n'''(c_0)) - f(c_0,m(c_0),m'(c_0),n''(c_0),n'''(c_0)) \\ &\quad + f(c_0,m(c_0),m'(c_0),n''(c_0),m'''(c_0)) - f(c_0,m(c_0),m'(c_0),m''(c_0),m'''(c_0)) \\ &\geqslant l\,[n''(c_0)-m''(c_0)] > 0,\end{aligned}$$

矛盾.

(2) 如果 $c_0=0$,则 $r(0)>0, r'(0) \leqslant 0$,则由(3.14),$(H_8)$,得

$$r(0) \leqslant r(0)-k_1 r'(0) \leqslant \int_0^1 [g_1(n''(s))-g_1(m''(s))]\,ds \leqslant \int_0^1 g'_1(\eta(s))\,r(s)\,ds < r(0),$$

矛盾.

(3) 如果 $c_0=1$,则 $r(1)>0, r'(1) \geqslant 0$,则由(3.14),$(H_8)$ 得

$$r(1)+k_2 r'(1) \leqslant \int_0^1 [g_2(n''(s))-g_1(m''(s))]\,ds \leqslant \int_0^1 g'_1(\eta(s))\,r(s)\,ds < r(1),$$

矛盾.

第二步,方程(1.2) 至少有一解 $u(t)$ 满足 $m(t) \leqslant u(t) \leqslant n(t), m'(t) \leqslant u'(t) \leqslant n'(t), n''(t) \leqslant u''(t) \leqslant m''(t), |u'''(t)| \leqslant N, t \in [0,1]$,其中 N 为与 m,n,Φ_H 有关的. 证明方法与定理 3.1 相同.

第三步,方程(1.2) 有唯一解. 假设 $x(t), y(t)$ 为方程(1.2) 两个不同解. 则 $x(t), y(t)$ 为方程(1.2) 的上下解. 由第一步得 $x(t) \leqslant y(t)$,而 $y(t)$,$x(t)$ 也为方程(1.2) 的上下解,由第一步得 $y(t) \leqslant x(t)$. 所以 $x(t)=y(t)$. 证毕.

4　例　子

考虑积分边界条件的四创边值问题

$$\begin{cases} u''' = u^{2m+1} + (u')^{2n+1} + 3(u''')^{2p+1}(|u'''|+1)^{\theta} + e(t), & 0<t<1, \\ u(0) = u'(1) = 0, \\ u''(0) - k_1 u'''(0) = \int_0^1 \left[-u(s) - \dfrac{\arctan(u'(s))}{\pi} + \dfrac{(u''(s))^3}{8}\right] ds, \\ u''(1) + k_2 u'''(1) = \int_0^1 \left[-u'(s) + (u''(s))^3\right] ds. \end{cases} \tag{4.1}$$

这里 $m,n,p \in N \cup \{0\}, \theta \in [0,2], e(t) \in C([0,1])$ 且 $\|e\| \leqslant \dfrac{3}{2}$.

$$f(t,x_0,x_1,x_2,x_3) = x_0^{2m+1} + (x_1)^{2n+1} + 3(x_2)^{2p+1} \times (|x_3|+1)^{\theta} + e(t),$$

$$h_1(x_0,x_1,x_2) = -x_0 - \frac{\arctan(x_1)}{\pi} + \frac{(x_2)^3}{8},$$

$$h_2(x_0,x_1,x_2) = -x_1 + \frac{(x_2)^3}{4}.$$

令 $\alpha,\beta: [0,1] \to R$ 为 $\alpha(t) = -\dfrac{1}{2}t(2-t)$,$\beta(t) = \dfrac{1}{2}t(2-t)$,则

$$f(t,\alpha(t),\alpha'(t),\alpha''(t),\alpha'''(t)) \geqslant -(1/2)^{2m+1} - (1-t)^{2n+1} + 3 + e(t) \geqslant 0 = \alpha'''(t),$$

$$f(t,\beta(t),\beta'(t),\beta''(t),\beta'''(t)) \leqslant (1/2)^{2m+1} + (1-t)^{2n+1} - 3 + e(t) \leqslant 0 = \alpha'''(t),$$

和(2.2) 成立. 所以 α,β 为(4.1) 的下上解,也就是 (H_1) 成立.

易证 $f(t,x_0,x_1,x_2,x_3)$ 在 E 上满足 Nagumo 条件且满足 (H_4),其中

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$E=\{(t,x_0,x_1,x_2,x_3)\in[0,1]\times R^4:\frac{t^2-2t}{2}\leqslant x\leqslant\frac{2t-t^2}{2},t-1\leqslant x_1\leqslant 1-t,-1\leqslant x_2\leqslant 1\}$.

(H_2),(H_5) 显然成立.

由定理 3.1 知,边值问题(4.1) 至少有一个解 $u(t)$ 且满足 $\frac{t^2-2t}{2}\leqslant u(t)\leqslant\frac{2t-t^2}{2},t-1\leqslant u'(t)\leqslant 1-t,-1\leqslant u''(t)\leqslant 1$.

参 考 文 献

[1] Bai Z B. The method of lower and upper solution for a bending of an elastic beam equation [J]. J. Math. Anal. Appl., 2000,248: 195 – 202.

[2] Bai Z B. The method of lower and upper solutions for some fourth-order boundary value problems [J]. Nonlinear Anal., 2007,67: 1704 – 1709.

[3] Chai G Q. Existence of positive solutions for foiurth-order boundary valne problem with variable parameters [J]. Nonlinear Anal. 2007,66: 870 – 880.

[4] Feng H Y, Ji D H, Ge W G. Existence and uniqueness of solutions for a fourth-order bound-ary value problem [J]. Nonlinear Anal.,2009,70: 3561 – 3566.

[5] Wang Y Y, Liu G F, Hu Y P. Existence and uniqueness of solutions for a second order differential equation with integral boundary conditions [J]. Applied Mathematics and Computation,2010,216: 2718 – 2727.

[6] Bai Z B. Positive solutions of some nonlocal fourth-order boundary value problem [J]. Applied Mathematics and Computation,2010,215: 4191 – 4197.

Existence and Uniqueness of Solutions for Fourth-Order *p*-Laplacian Differential Equations with Integral Boundary Conditions and a Sign-Changing Nonlinear Term

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Abstract

By the method of upper and lower solutions and Leary-Schauder degree theory, we investigate the existence and uniqueness of a solutions for the following four order differential equation

$$(\phi(u'''(t)))'-f(t,u(t),u'(t),u''(t),u'''(t))=0,t\in(0,1),$$

subject to the integral boundary conditions. $f:[0,1]\times R^4\to R$ are continuous and $\phi(u)$ is an increasing homeomorphism with $\phi(0)=0$, $\phi(R)=R$ where $R=(-\infty,+\infty)$.

Key words: intergral boundary conditions, upper and lower solutions, Leray Schauder degree theory, Nagumo condition, *p*-Laplacian.

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