DOI：10.19789/j.1004-9398.2023.03.001
文献引用：杨荣.对数Bloch类空间与n阶加权类空间之间的加权微分复合算子[J].首都师范大学学报（自然科学版），2023，44（3）：1-6. YANG R. Weighted differentiation composition operators between logarithmic Bloch type spaces and nth weighted type spaces[J]. Journal of Capital Normal University（Natural Science Edition），2023，44（3）：1-6.

对数Bloch类空间与n阶加权类空间之间的加权微分复合算子*

杨　荣**

（天津大学数学学院，天津　300350）

摘要：本文研究了从对数Bloch类空间$B^{\alpha}_{\log^{\beta}}$到$n$阶加权类空间$W^{(n)}_{\mu}$的加权微分复合算子$D^{m}_{\phi,u}$的有界性和紧性，同时当权函数$\mu(z)=v_{\alpha,\beta}(z)$时，也刻画了从$n$阶加权类空间$W^{(n)}_{v_{\alpha,\beta}}$到对数Bloch类空间$B^{\alpha}_{\log^{\beta}}$的加权微分复合算子$D^{m}_{\phi,u}$是有界和紧的充要条件。

关键词：加权微分复合算子；对数Bloch类空间；n阶加权类空间；有界性；紧性
中图分类号：O177.1　　**文献标识码**：A

Weighted differentiation composition operators between logarithmic Bloch type spaces and nth weighted type spaces*

YANG Rong**

（School of Mathematics，Tianjin University，Tianjin　300350）

Abstract: This paper characterizes the boundedness and compactness of weighted differentiation composition operators $D^{m}_{\phi,u}$ from logarithmic Bloch type spaces $B^{\alpha}_{\log^{\beta}}$ to nth weighted type spaces $W^{(n)}_{\mu}$. When $\mu(z)=v_{\alpha,\beta}(z)$, it is also showed that the necessary and sufficient conditions for the boundedness and compactness of the $D^{m}_{\phi,u}$ from $W^{(n)}_{v_{\alpha,\beta}}$ to $B^{\alpha}_{\log^{\beta}}$.

Keywords: weighted differentiation composition operator; logarithmic Bloch type spaces; nth weighted type spaces; boundedness; compactness
CLC: O177.1　　**DC**: A

0　引　言

对解析函数空间上不同算子基本性质的研究一直是算子理论中一个重要的课题，其中对加权微分复合算子性质的研究也是众多学者热衷的一个研究方向。加权微分复合算子首先是由Zhu[1]在2007年引进的，其在文章中刻画了从Bergman类空间到Bers类空间的加权微分复合算子的有界性和紧性。论文发表后引起了众多学者的关注，随后开始了对加权微分复合算子在单位圆盘上的加权Bergman空间、Hardy空间、Bloch类空间等多个空间上的有界性和紧性的研究，也进一步研究了不同空间之间的加权微分复合算子为有界算子和紧算子时的特性。Stevic[2]研究了从mixed-norm空间到加

收稿日期：2022-02-11
* 国家自然科学基金项目（12171353）
** 通信作者：1321860278@qq.com

权类空间的加权微分复合算子是有界算子和紧算子时的特性,紧接着他刻画了从H^∞空间和Bloch空间到n阶加权类空间的加权微分复合算子有界和紧的充要条件[3],并在2009年提出对数Bloch类空间的概念[4]。Esmaeili[5]研究了从对数Bloch类空间到n阶加权类空间的加权微分复合算子的有界性,且对其本性范数进行了估计。此外,文献[6-13]也对加权微分复合算子的性质进行了刻画。在文献[14-17]中对特殊的加权微分复合算子即加权复合算子的有界性和紧性进行了研究。可以看到在对加权微分复合算子研究的过程中,通常会先构造一个辅助函数,再利用线性组合构造新的辅助函数对所得的结论进行证明,在Li和Stevic[18-20]的论文中广为应用并形成了专门的方法。

本文在文献[5]的基础上,完整地刻画了从对数Bloch类空间$B^\alpha_{\log^\beta}$到n阶加权类空间$W^{(n)}_\mu$的加权微分复合算子$D^m_{\phi,u}$的有界性和紧性;当n阶加权类空间$W^{(n)}_\mu$的权函数$\mu(z)=v_{\alpha,\beta}(z)$时,也分别刻画了从$n$阶加权类空间$W^{(n)}_{v_{\alpha,\beta}}$到对数Bloch类空间$B^\alpha_{\log^\beta}$的加权微分复合算子$D^m_{\phi,u}$是有界和紧的充要条件。

1 预备知识

用$H(D)$表示复平面C中单位开圆盘D上的解析函数构成的集合,$S(D)$表示D上的解析自映射构成的集合。解析函数关于权函数v构成的加权Banach空间用H^∞_v表示,

$$H^\infty_v=\left\{f\in H(D):\|f\|_v=\sup_{z\in D}v(z)\left|f(z)\right|<\infty\right\}。$$

定义1.1 考虑权

$$v_{\alpha,\beta}(z):=\left(1-|z|^2\right)^\alpha\left(\ln\frac{\mathrm{e}^{\frac{\beta}{\alpha}}}{1-|z|^2}\right)^\beta,z\in D,$$

式中:$\alpha>0,\beta\geqslant 0$。

对数Bloch类空间的定义为

$$B^\alpha_{\log^\beta}=\left\{f\in H(D):\|f'\|_{v_{\alpha,\beta}}=\sup_{z\in D}v_{\alpha,\beta}(z)\left|f'(z)\right|<\infty\right\}。$$

$B^\alpha_{\log^\beta}$在范数$\|f\|_{B^\alpha_{\log^\beta}}=|f(0)|+\|f'\|_{v_{\alpha,\beta}}$下是一个Banach空间。

令$B^{\alpha,0}_{\log^\beta}$表示$B^\alpha_{\log^\beta}$的子空间,且这个子空间中的函数$f$满足

$$\lim_{|z|\to 1}v_{\alpha,\beta}(z)\left|f'(z)\right|=0。$$

定义1.2 对任意的一个权μ和一个非负整数n,n阶加权类空间$W^{(n)}_\mu$的定义为

$$W^{(n)}_\mu=\left\{f\in H(D):\sup_{z\in D}\mu(z)\left|f^{(n)}(z)\right|<\infty\right\}。$$

$W^{(n)}_\mu$在范数$\|f\|_{W^{(n)}_\mu}=\sum_{j=0}^{n-1}\left|f^{(j)}(0)\right|+\left\|f^{(n)}\right\|_\mu$下是一个Banach空间。当$n=0$时,$W^{(n)}_\mu$即为空间$H^\infty_\mu$;当$n=1$时,$W^{(n)}_\mu$为加权Bloch空间$B^\mu$;当$n=2$时,$W^{(n)}_\mu$为加权Zygmund空间$Z_\mu$。

令$W^{(n)}_{\mu,0}$表示$W^{(n)}_\mu$的子空间,且这个子空间中的函数f满足

$$\lim_{|z|\to 1}\mu(z)\left|f^{(n)}(z)\right|=0。$$

定义1.3 令m表示一个非负整数,$u\in H(D)$,$\phi\in S(D)$,则加权微分复合算子$D^m_{\phi,u}$的定义为$\left(D^m_{\phi,u}f\right)(z)=u(z)\cdot f^{(m)}(\phi(z)),f\in H(D),z\in D$。当$m=0$且$u(z)\equiv 1$时,$D^m_{\phi,u}$就是复合算子$C_\phi$;当$m=0$时,$D^m_{\phi,u}$就是加权复合算子$uC_\phi$,即是乘法算子$M_u$和复合算子$C_\phi$的乘积;当$m=1,u(z)=\phi'(z)$时,$D^m_{\phi,u}$就是微分算子与复合算子的乘积。

引理1.1[3] 设$n\in N_0,g,u\in H(D),\phi\in S(D)$。则

$$(u(z)g(\phi(z)))^{(n)}=\sum_{k=0}^{n}g^{(k)}(\phi(z))\cdot\sum_{l=k}^{n}C^n_l u^{(n-l)}(z)B_{l,k}(\phi'(z),\cdots,\phi^{(l-k+1)}(z)),$$

式中:$k=k_1+k_2+\cdots+k_l,k_1+2k_2+\cdots+lk_l=l$,且$k_j\geqslant 0,j=1,2,\cdots,l$,

$$B_{l,k}(\phi'(z),\cdots,\phi^{(l-k+1)}(z))=\sum_{k_1,k_2,\cdots,k_l}\frac{l!}{k_1!\ k_2!\ \cdots k_l!}\prod_{j=1}^{l}\left(\frac{\phi^{(j)}(z)}{j!}\right)^{k_j}。$$

特别地,$B_{n,n}(\phi'(z))=(\phi'(z))^n$,$B_{n,n-1}(\phi'(z),\phi''(z))=\frac{1}{2}n(n-1)(\phi'(z))^{n-2}\phi''(z)$。

由文献[5],为了方便,设$\Phi^{u,\phi}_{k,n}(z)=C^n_l u^{(n-l)}(z)\cdot B_{l,k}(\phi'(z),\cdots,\phi^{(l-k+1)}(z))$,式中$k\in\{0,\cdots,n\}$。则有

$$(u(z)g(\phi(z)))^{(n)}=\sum_{k=0}^{n}g^{(k)}(\phi(z))\Phi^{u,\phi}_{k,n}(z)。$$

与文献[21]中引理2.1的证明类似,可以得到下面的结果。

引理1.2 令$\alpha>0,\beta\geqslant 0,m\in N_0$使得$m\geqslant n$。设$f\in W^{(n)}_{v_{\alpha,\beta}}$,则存在一个常数$C$,使得

$$\left|f^{(m)}(z)\right|\leqslant\frac{C\|f\|_{W^{(n)}_{v_{\alpha,\beta}}}}{v_{\alpha+m-n,\beta}(z)},z\in D。$$

引理 1.3　令 $u \in H(D)$, $\phi \in S(D)$, 设 $\alpha > 0$, $\beta \geqslant 0$, $m \in N_0$ 使得 $m \geqslant n$, $\alpha - n + 1 > 0$。则算子 $D^m_{\phi,u}: W^{(n)}_{v_{\alpha,\beta}} \to B^{\alpha}_{\log^{\beta}}$ 是紧的当且仅当 $D^m_{\phi,u}: W^{(n)}_{v_{\alpha,\beta}} \to B^{\alpha}_{\log^{\beta}}$ 有界,且对于在 D 的紧子集上一致收敛于0的 $W^{(n)}_{v_{\alpha,\beta}}$ 中的任意有界序列 $\{f_k\}_{k\in N}$, 有 $\left\| D^m_{\phi,u} f_k \right\|_{B^{\alpha}_{\log^{\beta}}} \to 0, k \to \infty$。

这个引理的证明与文献[22]中命题3.11的证明类似,下面这个引理可由文献[23]中引理1的证明类似可得。

引理 1.4　设 $\alpha > 0, \beta \geqslant 0$, 则 $B^{\alpha,0}_{\log^{\beta}}$ 中的一个闭集 K 是紧的当且仅当其有界,且满足

$$\lim_{|z|\to 1} \sup_{f\in K} v_{\alpha,\beta}(z) \left| f'(z) \right| = 0。$$

2　主要结果

定理 2.1　令 $u \in H(D)$, $\phi \in S(D)$, μ 是任意的一个权,设 $\alpha > 0, \beta \geqslant 0, m \in N_0$ 使得 $\alpha + m > 1$。则下列表述等价:

(1) $D^m_{\phi,u}: B^{\alpha}_{\log^{\beta}} \to W^{(n)}_{\mu}$ 有界;

(2) $D^m_{\phi,u}: B^{\alpha,0}_{\log^{\beta}} \to W^{(n)}_{\mu}$ 有界;

(3) 对 $\forall k \in \{0, \cdots, n\}$, $\Phi^{u,\phi}_{k,n} \cdot C_{\phi}: H^{\infty}_{v_{\alpha+m+k-1,\beta}} \to H^{\infty}_{\mu}$ 有界;

(4) 对 $\forall k \in \{0, \cdots, n\}$, $\sup\limits_{z\in D} \dfrac{\mu(z)\left|\Phi^{u,\phi}_{k,n}(z)\right|}{v_{\alpha+m+k-1,\beta}(\phi(z))} < \infty$。

证明　由文献[5]的定理2.5可知(1) ⇔ (3) ⇔ (4),且(1) ⇒ (2)是显然的,而(2) ⇒ (4)与文献[4]的定理2.5的证明类似。

下面这个推论的证明与文献[3]中定理2的证明类似。

推论 2.1　令 $u \in H(D)$, $\phi \in S(D)$, μ 是任意的一个权,设 $\alpha > 0, \beta \geqslant 0, m \in N_0$ 使得 $\alpha + m > 1$。则 $D^m_{\phi,u}: B^{\alpha,0}_{\log^{\beta}} \to W^{(n)}_{\mu,0}$ 有界当且仅当 $D^m_{\phi,u}: B^{\alpha,0}_{\log^{\beta}} \to W^{(n)}_{\mu}$ 有界,且对 $\forall k \in \{0, \cdots, n\}$, $\lim\limits_{|z|\to 1} \mu(z)\left|\Phi^{u,\phi}_{k,n}(z)\right| = 0$。

定理 2.2　令 $u \in H(D)$, $\phi \in S(D)$, μ 是任意的一个权,设 $\alpha > 0, \beta \geqslant 0, m \in N_0$ 使得 $\alpha + m > 1$。则下列表述等价:

(1) $D^m_{\phi,u}: B^{\alpha}_{\log^{\beta}} \to W^{(n)}_{\mu}$ 紧;

(2) $D^m_{\phi,u}: B^{\alpha,0}_{\log^{\beta}} \to W^{(n)}_{\mu}$ 紧;

(3) 对 $\forall k \in \{0, \cdots, n\}$, $\Phi^{u,\phi}_{k,n} \cdot C_{\phi}: H^{\infty}_{v_{\alpha+m+k-1,\beta}} \to H^{\infty}_{\mu}$ 紧;

(4) $D^m_{\phi,u}: B^{\alpha}_{\log^{\beta}} \to W^{(n)}_{\mu}$ 有界,且对 $\forall k \in \{0, \cdots, n\}$, $\limsup\limits_{|\phi(z)|\to 1} \dfrac{\mu(z)\left|\Phi^{u,\phi}_{k,n}(z)\right|}{v_{\alpha+m+k-1,\beta}(\phi(z))} = 0$。

证明　由文献[5]可得(1) ⇔ (3) ⇔ (4),且(1) ⇒ (2)是显然的,与文献[3]中定理3的证明类似可得(2) ⇒ (4)。

定理 2.3　令 $u \in H(D)$, $\phi \in S(D)$。设 $\alpha > 0$, $\beta \geqslant 0, m \in N_0$ 使得 $m \geqslant n, \alpha - n + 1 > 0$。则下列表述等价:

(1) $D^m_{\phi,u}: W^{(n)}_{v_{\alpha,\beta}} \to B^{\alpha}_{\log^{\beta}}$ 有界;

(2) $\sup\limits_{z\in D} \dfrac{v_{\alpha,\beta}(z)\left|u'(z)\right|}{v_{\alpha+m-n,\beta}(\phi(z))} < \infty$,

$\sup\limits_{z\in D} \dfrac{v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|}{v_{\alpha+m+1-n,\beta}(\phi(z))} < \infty$。

证明　假设(2)成立,对 $\forall z \in D, f \in W^{(n)}_{v_{\alpha,\beta}}$, 有

$$\left\|(D^m_{\phi,u}f)'\right\|_{v_{\alpha,\beta}} = \sup_{z\in D} v_{\alpha,\beta}(z)\left|\ (u(z)f^{(m)}(\phi(z)))'\right| \leqslant$$

$$\sup_{z\in D} v_{\alpha,\beta}(z)\left|u'(z)f^{(m)}(\phi(z))\right| +$$

$$\sup_{z\in D} v_{\alpha,\beta}(z)\left|u(z)f^{(m+1)}(\phi(z))\phi'(z)\right| \leqslant$$

$$C\|f\|_{W^{(n)}_{v_{\alpha,\beta}}}\left(\sup_{z\in D}\frac{v_{\alpha,\beta}(z)\left|u'(z)\right|}{v_{\alpha+m-n,\beta}(\phi(z))} + \sup_{z\in D}\frac{v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|}{v_{\alpha+m+1-n,\beta}(\phi(z))}\right) < C\|f\|_{W^{(n)}_{v_{\alpha,\beta}}}。$$

另一方面,有

$$\left|\ (D^m_{\phi,u}f)(0)\right| = \left|u(0)f^{(m)}(\phi(0))\right| \leqslant C\|f\|_{W^{(n)}_{v_{\alpha,\beta}}} \cdot \frac{\left|u(0)\right|}{v_{\alpha+m-n,\beta}(\phi(0))}。$$

则可得 $D^m_{\phi,u}: W^{(n)}_{v_{\alpha,\beta}} \to B^{\alpha}_{\log^{\beta}}$ 有界。

反之,设 $D^m_{\phi,u}: W^{(n)}_{v_{\alpha,\beta}} \to B^{\alpha}_{\log^{\beta}}$ 有界。对 $\forall z \in D$, 考虑函数

$$g_z(\omega) = \sum_{j=1}^{m-n+2} \frac{a_j K^{n+j,\alpha+j}_z(\omega)}{\prod\limits_{l=0}^{n-1}(\alpha+j+l)}\left(\ln\frac{\mathrm{e}^{\frac{\beta}{\alpha}}}{1-\left|\phi(z)\right|^2}\right)^{-\beta} =$$

$$\sum_{j=1}^{m-n+2} \frac{a_j}{\prod\limits_{l=0}^{n-1}(\alpha+j+l)} \frac{\left(1-\left|\phi(z)\right|^2\right)^{n+j}}{\left(1-\overline{\phi(z)}\omega\right)^{\alpha+j}}\left(\ln\frac{\mathrm{e}^{\frac{\beta}{\alpha}}}{1-\left|\phi(z)\right|^2}\right)^{-\beta},$$

$$\omega \in D,$$

利用文献[3]中定理1的证明,可选择常数 $a_1, \cdots, a_{m-n+2}$ 使得

$$g_z^{(n+k)}(\phi(z))=\frac{\overline{\phi(z)}^{n+k}}{v_{\alpha+k,\beta}(\phi(z))},$$

和

$$g_z^{(n+t)}(\phi(z))=0,t\in\{0,\cdots,m-n+1\}\setminus\{k\}。$$

由文献[5]可知,$f_{z,j}=K_z^{n+j,\alpha+j}\left(\ln\frac{e^{\frac{\beta}{\alpha}}}{1-|\phi(z)|^2}\right)^{-\beta}\in B^{\alpha-n+1}_{\log^\beta},\forall j\in\{1,\cdots,m-n+2\}$。又由文献[5]的引理2.4有$f_{z,j}^{(n-1)}\in B^{\alpha}_{\log^\beta}$,则有$f_{z,j}\in W^{(n)}_{v_{\alpha,\beta}}$,所以$g_z\in W^{(n)}_{v_{\alpha,\beta}}$。

当$n+k=m$时,有

$$\frac{v_{\alpha,\beta}(z)|u'(z)||\phi(z)|^m}{v_{\alpha+m-n,\beta}(\phi(z))}\leqslant\left\|D^m_{\phi,u}g_z\right\|_{B^\alpha_{\log^\beta}},$$

可得

$$\sup_{|\phi(z)|>\frac{1}{2}}\frac{v_{\alpha,\beta}(z)|u'(z)|}{v_{\alpha+m-n,\beta}(\phi(z))}<\infty。\tag{2.1}$$

令$f_m(z)=\frac{z^m}{m!}\in W^{(n)}_{v_{\alpha,\beta}}$,则

$$\left\|(D^m_{\phi,u}f_m)'\right\|_{v_{\alpha,\beta}}=\sup_{z\in D}v_{\alpha,\beta}(z)\left|\left(u(z)f_m^{(m)}(\phi(z))\right)'\right|=\sup_{z\in D}v_{\alpha,\beta}(z)|u'(z)|<\infty,\tag{2.2}$$

所以

$$\sup_{|\phi(z)|\leqslant\frac{1}{2}}\frac{v_{\alpha,\beta}(z)|u'(z)|}{v_{\alpha+m-n,\beta}(\phi(z))}\leqslant\frac{\sup_{z\in D}v_{\alpha,\beta}(z)|u'(z)|}{v_{\alpha+m-n,\beta}\left(\frac{1}{2}\right)}<\infty。\tag{2.3}$$

由式(2.1)和(2.3)可得

$$\sup_{z\in D}\frac{v_{\alpha,\beta}(z)|u'(z)|}{v_{\alpha+m-n,\beta}(\phi(z))}<\infty。$$

同理,当$n+k=m+1$时,有

$$\sup_{|\phi(z)|>\frac{1}{2}}\frac{v_{\alpha,\beta}(z)|u(z)\phi'(z)|}{v_{\alpha+m+1-n,\beta}(\phi(z))}<\infty。$$

令$f_{m+1}(z)=\frac{z^{m+1}}{(m+1)!}\in W^{(n)}_{v_{\alpha,\beta}}$,则

$$\left\|(D^m_{\phi,u}f_{m+1})'\right\|_{v_{\alpha,\beta}}=\sup_{z\in D}v_{\alpha,\beta}(z)\left|\left(u(z)f_{m+1}^{(m)}(\phi(z))\right)'\right|=\sup_{z\in D}v_{\alpha,\beta}(z)|u'(z)\phi(z)+u(z)\phi'(z)|$$

可得

$$\sup_{z\in D}v_{\alpha,\beta}(z)|u(z)\phi'(z)|\leqslant\sup_{z\in D}v_{\alpha,\beta}(z)|u'(z)|+\left\|(D^m_{\phi,u}f_{m+1})'\right\|_{v_{\alpha,\beta}}\leqslant\sup_{z\in D}v_{\alpha,\beta}(z)|u'(z)|+\left\|D^m_{\phi,u}f_{m+1}\right\|_{B^\alpha_{\log^\beta}},$$

由式(2.2)和算子$D^m_{\phi,u}$有界可知

$$\sup_{z\in D}v_{\alpha,\beta}(z)|u(z)\phi'(z)|<\infty。$$

与式(2.3)类似,有

$$\sup_{|\phi(z)|\leqslant\frac{1}{2}}\frac{v_{\alpha,\beta}(z)|u(z)\phi'(z)|}{v_{\alpha+m+1-n,\beta}(\phi(z))}<\infty,$$

所以

$$\sup_{z\in D}\frac{v_{\alpha,\beta}(z)|u(z)\phi'(z)|}{v_{\alpha+m+1-n,\beta}(\phi(z))}<\infty。$$

定理2.4 令$u\in H(D),\phi\in S(D)$。设$\alpha>0,\beta\geqslant0,m\in N_0$使得$m\geqslant n,\alpha-n+1>0$。则下列表述等价:

(1)$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$紧;

(2)$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$有界,且

$$\lim_{|\phi(z)|\to1}\frac{v_{\alpha,\beta}(z)|u'(z)|}{v_{\alpha+m-n,\beta}(\phi(z))}=0,$$

$$\lim_{|\phi(z)|\to1}\frac{v_{\alpha,\beta}(z)|u(z)\phi'(z)|}{v_{\alpha+m+1-n,\beta}(\phi(z))}=0。$$

证明 设(2)成立,令$\{f_k\}_{k\in N}$是$W^{(n)}_{v_{\alpha,\beta}}$上的任意有界序列,且$\{f_k\}_{k\in N}$在$D$的紧子集上一致收敛于0。设存在常数$M$,使得$\sup_{k\in N}\|f_k\|_{W^{(n)}_{v_{\alpha,\beta}}}\leqslant M$。由柯西积分公式可知$f_k^{(m)},f_k^{(m+1)}$在$D$的紧子集上也一致收敛于0。

由假设,对$\forall\varepsilon>0$,存在常数$\delta\in(0,1)$,使得$\delta<|\phi(z)|<1$时有

$$\frac{v_{\alpha,\beta}(z)|u'(z)|}{v_{\alpha+m-n,\beta}(\phi(z))}<\frac{\varepsilon}{2M},\frac{v_{\alpha,\beta}(z)|u(z)\phi'(z)|}{v_{\alpha+m+1-n,\beta}(\phi(z))}<\frac{\varepsilon}{2M}。$$

因$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$有界,由权函数的有界性可得

$$\sup_{z\in D}v_{\alpha,\beta}(z)|u'(z)|<\infty,\sup_{z\in D}v_{\alpha,\beta}(z)|u(z)\phi'(z)|<\infty,\tag{2.4}$$

则

$$\left\|D^m_{\phi,u}f_k\right\|_{B^\alpha_{\log^\beta}}=|u(0)f_k^{(m)}(\phi(0))|+\sup_{z\in D}v_{\alpha,\beta}(z)\left|\left(u(z)f_k^{(m)}(\phi(z))\right)'\right|\leqslant|u(0)||f_k^{(m)}(\phi(0))|+\sup_{z\in D}v_{\alpha,\beta}(z)|u'(z)||f_k^{(m)}(\phi(z))|+\sup_{z\in D}v_{\alpha,\beta}(z)|u(z)\phi'(z)||f_k^{(m+1)}(\phi(z))|\leqslant|u(0)||f_k^{(m)}(\phi(0))|+$$

$$\sup_{|\phi(z)|\leqslant\delta} v_{\alpha,\beta}(z)\left|u'(z)\right|\left|f_k^{(m)}(\phi(z))\right|+$$
$$\sup_{|\phi(z)|\leqslant\delta} v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|\left|f_k^{(m+1)}(\phi(z))\right|+$$
$$\sup_{\delta<|\phi(z)|<1} v_{\alpha,\beta}(z)\left|u'(z)\right|\left|f_k^{(m)}(\phi(z))\right|+$$
$$\sup_{\delta<|\phi(z)|<1} v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|\left|f_k^{(m+1)}(\phi(z))\right|\leqslant$$
$$\left|u(0)\right|\left|f_k^{(m)}(\phi(0))\right|+$$
$$\sup_{z\in D} v_{\alpha,\beta}(z)\left|u'(z)\right|\sup_{|\omega|\leqslant\delta}\left|f_k^{(m)}(\omega)\right|+$$
$$\sup_{z\in D} v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|\sup_{|\omega|\leqslant\delta}\left|f_k^{(m+1)}(\omega)\right|+C\left\|f_k\right\|_{W^{(n)}_{v_{\alpha,\beta}}}\cdot$$
$$\left(\sup_{\delta<|\phi(z)|<1}\frac{v_{\alpha,\beta}(z)\left|u'(z)\right|}{v_{\alpha+m-n,\beta}(\phi(z))}+\right.$$
$$\left.\sup_{\delta<|\phi(z)|<1}\frac{v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|}{v_{\alpha+m+1-n,\beta}(\phi(z))}\right)<$$
$$\left|u(0)\right|\left|f_k^{(m)}(\phi(0))\right|+$$
$$\sup_{z\in D} v_{\alpha,\beta}(z)\left|u'(z)\right|\sup_{|\omega|\leqslant\delta}\left|f_k^{(m)}(\omega)\right|+$$
$$\sup_{z\in D} v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|\sup_{|\omega|\leqslant\delta}\left|f_k^{(m+1)}(\omega)\right|+C\varepsilon。$$

由ε的任意性和式(2.4)可得$\left\|D^m_{\phi,u}f_k\right\|_{B^\alpha_{\log^\beta}}\to 0,\ k\to\infty$。所以由引理1.3有$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$紧。

反之,设算子$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$紧,则$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$有界。令$\{z_i\}_{i\in\mathbf{N}}$是$D$中的一个序列,使得$|\phi(z_i)|\to 1,i\to\infty$。令$\{g_{z_i}\}_{i\in\mathbf{N}}$为定理2.3中的形式,由文献[5]可知$\{g_{z_i}\}_{i\in\mathbf{N}}$在$W^{(n)}_{v_{\alpha,\beta}}$中有界,且$g_{z_i}$在$D$的紧子集上一致收敛于0。

因为$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$紧,则

$$\lim_{i\to\infty}\left\|D^m_{\phi,u}g_{z_i}\right\|_{B^\alpha_{\log^\beta}}=0。\tag{2.5}$$

当$n+k=m$时,由定理2.3的证明有

$$\left\|D^m_{\phi,u}g_{z_i}\right\|_{B^\alpha_{\log^\beta}}\geqslant\frac{v_{\alpha,\beta}(z_i)\left|u'(z_i)\right|\left|\phi(z_i)\right|^m}{v_{\alpha+m-n,\beta}(\phi(z_i))}。$$

由$|\phi(z_i)|\to 1,i\to\infty$和式(2.5)可得

$$\lim_{i\to\infty}\frac{v_{\alpha,\beta}(z_i)\left|u'(z_i)\right|}{v_{\alpha+m-n,\beta}(\phi(z_i))}=0。$$

即有

$$\lim_{|\phi(z)|\to 1}\frac{v_{\alpha,\beta}(z)\left|u'(z)\right|}{v_{\alpha+m-n,\beta}(\phi(z))}=0。$$

同理,当$n+k=m+1$时,有

$$\left\|D^m_{\phi,u}g_{z_i}\right\|_{B^\alpha_{\log^\beta}}\geqslant\frac{v_{\alpha,\beta}(z_i)\left|u(z_i)\phi'(z_i)\right|\left|\phi(z_i)\right|^{m+1}}{v_{\alpha+m+1-n,\beta}(\phi(z_i))},$$

与$n+k=m$时的证明类似可得

$$\lim_{|\phi(z)|\to 1}\frac{v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|}{v_{\alpha+m+1-n,\beta}(\phi(z))}=0。$$

定理2.5　令$u\in H(D),\phi\in S(D)$。设$\alpha>0,\beta\geqslant 0,m\in N_0$使得$m\geqslant n,\alpha-n+1>0$。则下列表述等价:

(1) $D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^{\alpha,0}_{\log^\beta}$紧;

(2) $D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^{\alpha,0}_{\log^\beta}$有界;

(3) $D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$有界,且

$$\lim_{|z|\to 1}\frac{v_{\alpha,\beta}(z)\left|u'(z)\right|}{v_{\alpha+m-n,\beta}(\phi(z))}=0,\lim_{|z|\to 1}\frac{v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|}{v_{\alpha+m+1-n,\beta}(\phi(z))}=0。$$

证明　(1)⇒(2)显然,与定理2.3的证明类似可得(2)⇔(3)。

设(3)成立,则

$$\lim_{|z|\to 1}\sup_{\|f\|_{W^{(n)}_{v_{\alpha,\beta}}}\leqslant 1} v_{\alpha,\beta}(z)\left|\left(D^m_{\phi,u}f\right)'(z)\right|=$$
$$\lim_{|z|\to 1}\sup_{\|f\|_{W^{(n)}_{v_{\alpha,\beta}}}\leqslant 1} v_{\alpha,\beta}(z)\left|\left(u(z)f^{(m)}(\phi(z))\right)'\right|\leqslant$$
$$\lim_{|z|\to 1}\sup_{\|f\|_{W^{(n)}_{v_{\alpha,\beta}}}\leqslant 1} v_{\alpha,\beta}(z)\left|u'(z)\right|\left|f^{(m)}(\phi(z))\right|+$$
$$\lim_{|z|\to 1}\sup_{\|f\|_{W^{(n)}_{v_{\alpha,\beta}}}\leqslant 1} v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|\left|f^{(m+1)}(\phi(z))\right|\leqslant$$
$$C\lim_{|z|\to 1}\frac{v_{\alpha,\beta}(z)\left|u'(z)\right|}{v_{\alpha+m-n,\beta}(\phi(z))}+$$
$$C\lim_{|z|\to 1}\frac{v_{\alpha,\beta}(z)\left|u(z)\phi'(z)\right|}{v_{\alpha+m+1-n,\beta}(\phi(z))}=0。$$

则由引理1.4可知,$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^{\alpha,0}_{\log^\beta}$紧。

推论2.2　令$u\in H(D),\phi\in S(D)$。设$\alpha>0,\beta\geqslant 0,m\in N_0$使得$m\geqslant n,\alpha-n+1>0$。则当$D^m_{\phi,u}:W^{(n)}_{v_{\alpha,\beta}}\to B^\alpha_{\log^\beta}$(或$B^{\alpha,0}_{\log^\beta}$)有界时,有$u\in B^\alpha_{\log^\beta}$(或$B^{\alpha,0}_{\log^\beta}$)。

3　结束语

本文完整地刻画了从对数Bloch类空间$B^\alpha_{\log^\beta}$到n阶加权类空间$W^{(n)}_\mu$和从n阶加权类空间$W^{(n)}_{v_{\alpha,\beta}}$到对数Bloch类空间$B^\alpha_{\log^\beta}$的加权微分复合算子$D^m_{\phi,u}$是有界和紧的充要条件,且可以看到从$n$阶加权类空间$W^{(n)}_{v_{\alpha,\beta}}$到小对数Bloch类空间$B^{\alpha,0}_{\log^\beta}$的加权微分复合算子$D^m_{\phi,u}$的有界和紧是等价的。

参 考 文 献

[1] ZHU X L. Products of differentiation, composition and multiplication from Bergman type spaces to Bers type spaces[J]. Integral Transforms and Special Functions, 2007,18(3):223-231.

[2] STEVIC S. Weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces[J]. Applied Mathematics and Computation,2009, 211(1):222-233.

[3] STEVIC S. Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk[J]. Applied Mathematics and Computation,2010,216(12):3634-3641.

[4] STEVIC S. On new Bloch-type spaces[J]. Applied Mathematics and Computation,2009,215(2):841-849.

[5] ESMAEILI K. Generalized weighted composition operators from logarithmic Bloch type spaces to nth weighted type spaces[J]. Sahand Communications in Mathematical Analysis,2019,15(1):119-133.

[6] LIANG X Y, DONG T X. New characterizations for the products of differentiation and composition operators between Bloch-type spaces[J]. Journal of Inequalities and Applications,2014,502:14.

[7] OHNO S. Products of differentiation and composition on Bloch spaces[J]. Bulletin of the Korean Mathematical Society,2009,46(6):1135-1140.

[8] STEVIC S. Weighted differentiation composition operators from the mixed-norm space to the nth weighted-type space on the unit disk[J]. Abstract and Applied Analysis,2010,246287:15.

[9] STEVIC S. Products of composition and differentiation operators on the weighted Bergman space[J]. Bulletin of the Belgian Mathematical Society,2009,16(4):623-635.

[10] STEVIC S. Composition followed by differentiation from H^∞ and the Bloch space to nth weighted-type space on the unit disk [J]. Applied Mathematics and Computation,2010,216(12):3450-3458.

[11] ZHU X L. Generalized weighted composition operators on weighted Bergman spaces[J]. Numberical Functional Analysis and Optimization,2009,30(7):881-893.

[12] ZHU X L. A new characterization of the generalized weighted composition operator from H^∞ into the Zygmund space [J]. Mathematical Inequalities and Applications,2015,18(3):1135-1142.

[13] ZHU X L. Generalized weighted composition operators on Bloch-type spaces [J]. Journal Inequalities and Applications,2015,59:9.

[14] COLONNA F, HMIDOUCH N. Weighted composition operators between weighted-type spaces and the Bloch space and BMOA [J]. Advances in Operator Theory, 2020,5(1):94-114.

[15] COLONNA F, LI X S. Weighted composition operators from Hardy spaces into logarithmic Bloch spaces[J]. Journal of Function Spaces and Applications, 2012, 454820:20.

[16] MONTES-RODRIGUEZ A. Weighted composition operators on weighted Banach spaces of analytic functions [J]. Journal of the London Mathematical Society,2000,61(3):872-884.

[17] RAMOS-FERNANDEZ J C. Logarithmic Bloch spaces and their weighted composition operators[J]. Rendiconti del Circolo Matematico di Palermo, 2016, 65 (1) : 159-174.

[18] LI X S, STEVIC S. Composition followed by differentiation between Bloch type spaces[J]. Journal of Computational Analysis and Applications, 2007, 9(2): 195-205.

[19] LI X S, STEVIC S. Composition followed by differentiation between H^∞ and α-Bloch spaces [J]. Houston Journal of Mathematics,2009,35(1):327-340.

[20] LI X S, STEVIC S. Generalized weighted composition operators from α-Bloch spaces into weighted-type spaces [J]. Journal of Inequalities and Applications,2015,265:12.

[21] QU Y H,LIU M Y,CHENG L S. Weighted differentiation composition operator from logarithmic Bloch spaces to Zygmund-type spaces [J]. Abstract and Applied Analysis,2014,832713:14.

[22] COWEN C C, MACCLUER B D. Composition operators on spaces of analytic functions[M]. Boca Raton: CRC Press,1995.

[23] MADIGAN K, MATHESON A. Compact composition operators on the Bloch space [J]. Transactions of the American Mathematics Society,1995,347(7):2679-2687.

(责任编辑:马田田)