首都师范大学学报(自然科学版)
第 41 卷 第 3 期 Journal of Capital Normal University No. 3
2020 年 6 月 (Natural Science Edition) Jun. ,2020

$(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的 Hermite-Hadamard 型不等式*

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摘要:从$(\alpha,\beta,\lambda,\lambda_0,h)$凸函数的定义出发,用数学分析的方法建立了$(\alpha,\beta,\lambda,\lambda_0,h)$凸函数的 Hermite-Hadamard 型不等式. 在 h 可微且满足 $h(0)=0$, $h(1)=1$ 的情况下, 利用$(\alpha,\beta,\lambda,\lambda_0,h)$凸函数与其导数的关系,获得$(\alpha,\beta,\lambda,\lambda_0,h)$凸函数的 Hermite-Hadamard 型不等式.

关键词:凸函数; $(\alpha,\beta,\lambda,\lambda_0,h)$凸函数; Hermite-Hadamard 型不等式

中图分类号:O174.13; O178 **DOI**:10.19789/j.1004-9398.2020.03.001

0 引 言

作为 (α,h) 凸函数[1]、m-凸函数[2-3]、(h,m) 凸函数[4]、第一种意义上的(α,m)凸函数[5]、第二种意义上的(α,m)凸函数[6]等概念的推广,文献[7]引入了 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的概念.

定义 1[7] 设函数 $f:(0,+\infty)\to(0,+\infty)$, $h:[0,1]\to(0,+\infty)$,若存在 $\lambda_0\in(0,1]$ 及非零实数 α,β, 使得对任意 $x_1,x_2\in(0,\infty)$ 和任意 $\lambda\in[0,1]$, 有

$$f((\lambda x_1^{\alpha}+\lambda_0(1-\lambda)x_2^{\alpha})^{1/\alpha})\leqslant \{h(\lambda)f^{\beta}(x_1)+\lambda_0 h(1-\lambda)f^{\beta}(x_2)\}^{1/\beta}, \quad (1)$$

则称 f 是$(0,+\infty)$上的 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数.

当 $\beta>0(\beta<0)$ 时,式(1)等价于

$$f^{\beta}((\lambda x_1^{\alpha}+\lambda_0(1-\lambda)x_2^{\alpha})^{1/\alpha})\leqslant(\geqslant) h(\lambda)f^{\beta}(x_1)+\lambda_0 h(1-\lambda)f^{\beta}(x_2).$$

关于各类凸函数的 Hermite-Hadamard 型不等式可参考文献[3,8-14]. 本文目的是建立 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的 Hermite-Hadamard 型不等式.

收稿日期:2019-03-30

* 国家自然科学基金青年科学基金项目(11301090);广东省自然科学基金自由申请项目(2015A030313896);广东省特色创新项目(自然科学)(2016KTSCX094);广州市科学(技术)研究专项一般项目(201707010230);广东第二师范学院教授博士专项科研经费资助项目(2015ARF24)

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引理 1 设 $f:(0,+\infty)\to(0,+\infty)$是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数, $h(\lambda)$ 分别在点 $\lambda=0$ 和 $\lambda=1$ 处右可微和左可微,且 $h(0)=0$, $h(1)=1$, 则对任意 $x,y>0$, 且 f 在点 y 处可微,有

$$\frac{1}{\beta}[\lambda_0 h'_{+}(0)f^{\beta}(x)-h'_{-}(1)f^{\beta}(y)]\geqslant \frac{y^{1-\alpha}f^{\beta-1}(y)f'(y)}{\alpha}(\lambda_0 x^{\alpha}-y^{\alpha}). \quad (2)$$

证明 由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义,对任意 $\lambda\in[0,1)$, 有

$$f((\lambda y^{\alpha}+\lambda_0(1-\lambda)x^{\alpha})^{1/\alpha})\leqslant \{h(\lambda)f^{\beta}(y)+\lambda_0 h(1-\lambda)f^{\beta}(x)\}^{1/\beta},$$

由此得

$$\frac{f((\lambda y^{\alpha}+\lambda_0(1-\lambda)x^{\alpha})^{1/\alpha})-f(y)}{1-\lambda}\leqslant \frac{\{h(\lambda)f^{\beta}(y)+\lambda_0 h(1-\lambda)f^{\beta}(x)\}^{1/\beta}-f(y)}{1-\lambda}. \quad (3)$$

因为当 $\lambda\to1-0$ 时,

$$\{h(\lambda)f^{\beta}(y)+\lambda_0 h(1-\lambda)f^{\beta}(x)\}^{1/\beta}-f(y)=f(y)\cdot\left\{\exp\left[\frac{1}{\beta}\ln\frac{h(\lambda)f^{\beta}(y)+\lambda_0 h(1-\lambda)f^{\beta}(x)}{f^{\beta}(y)}\right]-1\right\}\sim \frac{f(y)}{\beta}\left[\frac{h(\lambda)f^{\beta}(y)+\lambda_0 h(1-\lambda)f^{\beta}(x)}{f^{\beta}(y)}-1\right]=\frac{f^{1-\beta}(y)}{\beta}\{[h(\lambda)-h(1)]f^{\beta}(y)+\lambda_0[h(1-\lambda)-h(0)]f^{\beta}(x)\},$$

所以式(3)右边的极限为

$$\lim_{\lambda\to 1-0}\frac{f^{1-\beta}(y)}{\beta}\left[\lambda_0 f^{\beta}(x)\,\frac{h(1-\lambda)-h(0)}{1-\lambda}-\frac{h(\lambda)-h(1)}{\lambda-1}f^{\beta}(y)\right]=$$

$$\frac{f^{1-\beta}(y)}{\beta}[\lambda_0 h'_{+}(0)f^{\beta}(x)-h'_{-}(1)f^{\beta}(y)],\quad(4)$$

若 $\lambda_0 x^{\alpha}=y^{\alpha}$, 则式(3)的左边为零. 在式(3)中,令 $\lambda\to 1-0$, 则得

$$\frac{f^{1-\beta}(y)}{\beta}[\lambda_0 h'_{+}(0)f^{\beta}(x)-h'_{-}(1)f^{\beta}(y)]\geqslant 0=\frac{y^{1-\alpha}f^{\beta-1}(y)f'(y)}{\alpha}(\lambda_0 x^{\alpha}-y^{\alpha}),$$

故式(2)成立. 若 $\lambda_0 x^{\alpha}\neq y^{\alpha}$, 则式(3)左边的极限为

$$\lim_{\lambda\to 1-0}\frac{f((\lambda y^{\alpha}+\lambda_0(1-\lambda)x^{\alpha})^{1/\alpha})-f(y)}{1-\lambda}=\lim_{\lambda\to 1-0}\frac{f((\lambda y^{\alpha}+\lambda_0(1-\lambda)x^{\alpha})^{1/\alpha})-f(y)}{(\lambda y^{\alpha}+\lambda_0(1-\lambda)x^{\alpha})^{1/\alpha}-y}\cdot\frac{(\lambda y^{\alpha}+\lambda_0(1-\lambda)x^{\alpha})^{1/\alpha}-y}{1-\lambda}=$$

$$-f'(y)\,\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda y^{\alpha}+\lambda_0(1-\lambda)x^{\alpha})^{1/\alpha}\Big|_{\lambda=1}=\frac{y^{1-\alpha}f'(y)}{\alpha}(\lambda_0 x^{\alpha}-y^{\alpha}).\quad(5)$$

在式(3)中,令 $\lambda\to 1-0$, 利用式(4)和式(5),则式(2)得证.

1 主要结果

定理 1 设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数, $\beta>0,a,b>0$, 且 $(2-\lambda)b^{\alpha}\neq\lambda_0\lambda a^{\alpha}$, f 在以 $(\lambda_0\lambda)^{1/\alpha}a,(2-\lambda)^{1/\alpha}b$ 为端点的区间上勒贝格可积,则有

$$f^{\beta}\left[\left(\frac{(2-\lambda)b^{\alpha}+\lambda_0\lambda a^{\alpha}}{2}\right)^{1/\alpha}\right]\leqslant\frac{\alpha h(1/2)}{(2-\lambda)b^{\alpha}-\lambda_0\lambda a^{\alpha}}\int_{(\lambda_0\lambda)^{1/\alpha}a}^{(2-\lambda)^{1/\alpha}b}x^{\alpha-1}\cdot[f^{\beta}(x)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}x)]\mathrm{d}x\leqslant h\left(\frac{1}{2}\right)\left[f^{\beta}((2-\lambda)^{1/\alpha}b)+2\lambda_0 f^{\beta}(\lambda^{1/\alpha}a)+\lambda_0^{2}f^{\beta}\left(\left(\frac{2-\lambda}{\lambda_0^{2}}\right)^{1/\alpha}b\right)\right].\quad(6)$$

证明 设

$$x(\lambda,t)=(t(2-\lambda)b^{\alpha}+(1-t)\lambda_0\lambda a^{\alpha})^{1/\alpha},$$

$$y(\lambda,t)=\left(t\lambda a^{\alpha}+(1-t)\frac{2-\lambda}{\lambda_0}b^{\alpha}\right)^{1/\alpha},$$

则有 $(x(\lambda,t))^{\alpha}+\lambda_0(y(\lambda,t))^{\alpha}=(2-\lambda)b^{\alpha}+\lambda_0\lambda a^{\alpha}$, 由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义有

$$f^{\beta}\left[\left(\frac{(2-\lambda)b^{\alpha}+\lambda_0\lambda a^{\alpha}}{2}\right)^{1/\alpha}\right]=f^{\beta}\left[\left(\frac{(x(\lambda,t))^{\alpha}+(y(\lambda,t))^{\alpha}}{2}\right)^{1/\alpha}\right]\leqslant h(1/2)f^{\beta}(x(\lambda,t))+\lambda_0 h(1/2)f^{\beta}(y(\lambda,t))\leqslant h(1/2)[h(t)f^{\beta}((2-\lambda)^{1/\alpha}b)+\lambda_0 h(1-t)f^{\beta}(\lambda^{1/\alpha}a)]+\lambda_0 h(1/2)\cdot\left[h(t)f^{\beta}(\lambda^{1/\alpha}a)+\lambda_0 h(1-t)f^{\beta}\left(\left(\frac{2-\lambda}{\lambda_0^{2}}\right)^{1/\alpha}b\right)\right],\quad(7)$$

在式(7)中对 t 在 $[0,1]$ 上积分,则式(6)得证.

推论 1 设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数, $\beta>0,a,b>0$, 且 $b^{\alpha}\neq\lambda_0 a^{\alpha}$, f 在以 $\lambda_0^{1/\alpha}a,b$ 为端点的区间上勒贝格可积,则有

$$f^{\beta}\left[\left(\frac{b^{\alpha}+\lambda_0 a^{\alpha}}{2}\right)^{1/\alpha}\right]\leqslant\frac{\alpha h(1/2)}{b^{\alpha}-\lambda_0 a^{\alpha}}\int_{\lambda_0^{1/\alpha}a}^{b}x^{\alpha-1}\cdot[f^{\beta}(x)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}x)]\mathrm{d}x\leqslant h(1/2)[f^{\beta}(b)+2\lambda_0 f^{\beta}(a)+\lambda_0^{2}f^{\beta}(\lambda_0^{-2/\alpha}b)].$$

定理 2 设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数, $\beta>0,0<a<b,f\in L[a,b],h\in L(0,1)$, 则有

$$\frac{1}{h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)}\cdot f^{\beta}\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^{\alpha}+b^{\alpha})\right)^{1/\alpha}\right]\leqslant\frac{1}{2(b-a)}\int_a^b[f^{\beta}(x)+f^{\beta}((a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha})]\mathrm{d}x\leqslant\frac{\min\{f^{\beta}(a)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}b),f^{\beta}(b)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}a)\}}{2(b-a)}\cdot\int_a^b\left[h\left(\frac{b^{\alpha}-x^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)+h\left(\frac{x^{\alpha}-a^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)\right]\mathrm{d}x.\quad(8)$$

证明 在式(1)中取 $\lambda=\frac{\lambda_0}{1+\lambda_0}$, 得

$$f^{\beta}\left[\left(\frac{\lambda_0}{1+\lambda_0}(x_1^{\alpha}+x_2^{\alpha})\right)^{1/\alpha}\right]\leqslant h\left(\frac{\lambda_0}{1+\lambda_0}\right)f^{\beta}(x_1)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)f^{\beta}(x_2),\quad(9)$$

在式(9)中取 $x_1=x,x_2=(a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha}$, 得

$$f^{\beta}\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^{\alpha}+b^{\alpha})\right)^{1/\alpha}\right]\leqslant h\left(\frac{\lambda_0}{1+\lambda_0}\right)f^{\beta}(x)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)f^{\beta}((a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha}),\quad(10)$$

在式(9)中取 $x_1=(a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha},x_2=x$, 得

$$f^{\beta}\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^{\alpha}+b^{\alpha})\right)^{1/\alpha}\right]\leqslant h\left(\frac{\lambda_0}{1+\lambda_0}\right)\cdot f^{\beta}((a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha})+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)f^{\beta}(x),\tag{11}$$

将式(10)和式(11)相加得

$$2f^{\beta}\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^{\alpha}+b^{\alpha})\right)^{1/\alpha}\right]\leqslant\left[h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)\right][f^{\beta}((a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha})+f^{\beta}(x)],\tag{12}$$

在式(12)中对 x 在$[a,b]$上求积分,则式(8)的左边部分得证.

对任意 $x\in[a,b]$,取 $t=\dfrac{b^{\alpha}-x^{\alpha}}{b^{\alpha}-a^{\alpha}}$,则有

$$x=(ta^{\alpha}+(1-t)b^{\alpha})^{1/\alpha},$$

$$(a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha}=((1-t)a^{\alpha}+tb^{\alpha})^{1/\alpha},$$

由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义

$$f^{\beta}(x)\leqslant\min\{h(t)f^{\beta}(a)+\lambda_0 h(1-t)f^{\beta}(\lambda_0^{-1/\alpha}b),\ h(1-t)f^{\beta}(b)+\lambda_0 h(t)f^{\beta}(\lambda_0^{-1/\alpha}a)\},\tag{13}$$

$$f^{\beta}((a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha})\leqslant\min\{h(1-t)f^{\beta}(a)+\lambda_0 h(t)f^{\beta}(\lambda_0^{-1/\alpha}b),\ h(t)f^{\beta}(b)+\lambda_0 h(1-t)f^{\beta}(\lambda_0^{-1/\alpha}a)\},\tag{14}$$

将式(13)和式(14)相加,得

$$f^{\beta}(x)+f^{\beta}((a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha})\leqslant\min\{[h(1-t)+h(t)][f^{\beta}(a)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}b)],\ [h(1-t)+h(t)][f^{\beta}(b)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}a)]\}=\left[h\left(\frac{b^{\alpha}-x^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)+h\left(\frac{x^{\alpha}-a^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)\right]\min\{f^{\beta}(a)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}b),f^{\beta}(b)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}a)\},\tag{15}$$

在式(15)中对 x 在$[a,b]$上求积分,则式(8)的右边部分得证.

定理 3　设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数,$\beta>0$,$0<a<b$,$g:[a,b]\to(0,+\infty)$ 是满足 $g((a^{\alpha}+b^{\alpha}-x^{\alpha})^{1/\alpha})=g(x)$ 的可积函数,$\int_a^b x^{\alpha-1}g(x)f^{\beta}(x)\mathrm{d}x<+\infty$,$h\in L(0,1)$,则有

$$\frac{\int_a^b x^{\alpha-1}g(x)\mathrm{d}x}{h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)}f^{\beta}\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^{\alpha}+b^{\alpha})\right)^{1/\alpha}\right]\leqslant\int_a^b x^{\alpha-1}g(x)f^{\beta}(x)\mathrm{d}x\leqslant\frac{\min\{f^{\beta}(a)+\lambda_0 f^{\beta}(b),f^{\beta}(b)+\lambda_0 f^{\beta}(a)\}}{2}\cdot\int_a^b x^{\alpha-1}g(x)\left[h\left(\frac{b^{\alpha}-x^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)+h\left(\frac{x^{\alpha}-a^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)\right]\mathrm{d}x.\tag{16}$$

证明　对任意 $t\in[0,1]$,令 $x_1(t)=(ta^{\alpha}+(1-t)b^{\alpha})^{1/\alpha}$,$x_2(t)=((1-t)a^{\alpha}+tb^{\alpha})^{1/\alpha}$,则有 $(x_1(t))^{\alpha}+(x_2(t))^{\alpha}=a^{\alpha}+b^{\alpha}$,由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义有

$$h\left(\frac{\lambda_0}{1+\lambda_0}\right)f^{\beta}(x_1(t))+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)f^{\beta}(x_2(t))\geqslant f^{\beta}\left\{\left(\frac{\lambda_0}{1+\lambda_0}[(x_1(t))^{\alpha}+(x_2(t))^{\alpha}]\right)^{1/\alpha}\right\}=f^{\beta}\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^{\alpha}+b^{\alpha})\right)^{1/\alpha}\right],\tag{17}$$

将式(17)乘以 $g(x_1(t))$,然后对 t 在$[0,1]$上积分,并注意到

$$\int_0^1 g(x_1(t))f^{\beta}(x_2(t))\mathrm{d}t=\int_0^1 g(x_1(t))f^{\beta}(x_1(t))\mathrm{d}t=\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b x^{\alpha-1}g(x)f^{\beta}(x)\mathrm{d}x,$$

$$\int_0^1 g(x_1(t))\mathrm{d}t=\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b x^{\alpha-1}g(x)\mathrm{d}x,$$

则式(16)的左边部分得证.由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义得

$$f^{\beta}(x_1(t))\leqslant\min\{h(t)f^{\beta}(a)+\lambda_0 h(1-t)f^{\beta}(\lambda_0^{-1/\alpha}b),\ h(1-t)f^{\beta}(b)+\lambda_0 h(t)f^{\beta}(\lambda_0^{-1/\alpha}a)\},\tag{18}$$

将式(18)乘以 $g(x_1(t))$,然后对 t 在$[0,1]$上积分,并注意到

$$\int_0^1 h(1-t)g(x_1(t))\mathrm{d}t=\int_0^1 h(1-t)g(x_1(1-t))\mathrm{d}t=\int_0^1 h(t)g(x_1(t))\mathrm{d}t=\frac{1}{2}\int_0^1[h(t)+h(1-t)]g(x_1(t))\mathrm{d}t=\frac{\alpha}{2(b^{\alpha}-a^{\alpha})}\int_a^b x^{\alpha-1}g(x)\left[h\left(\frac{b^{\alpha}-x^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)+h\left(\frac{x^{\alpha}-a^{\alpha}}{b^{\alpha}-a^{\alpha}}\right)\right]\mathrm{d}x,$$

则式(16)的右边部分得证.

推论 2　设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,$

$\lambda_0,h)$ 凸函数, $\beta>0,0<a<b,\int_a^b x^{\alpha-1}f^\beta(x)\mathrm{d}x<+\infty,h\in L(0,1)$, 则有

$$\frac{1}{h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)}f^\beta\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^\alpha+b^\alpha)\right)^{1/\alpha}\right]\leqslant$$

$$\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b x^{\alpha-1}f^\beta(x)\mathrm{d}x\leqslant\min\{f^\beta(a)+\lambda_0 f^\beta(b),$$

$$f^\beta(b)+\lambda_0 f^\beta(a)\}\int_0^1 h(t)\mathrm{d}t.$$

证明 在定理 3 中取 $g(x)\equiv 1$ 即可得证.

推论 3 设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数, $\beta>0,0<a<b,h\in L(0,1)$, $\int_a^b\frac{f^\beta(x)}{x(a^\alpha+b^\alpha-x^\alpha)}\mathrm{d}x<+\infty$, 则有

$$\frac{1}{h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)}f^\beta\left[\left(\frac{\lambda_0}{1+\lambda_0}(a^\alpha+b^\alpha)\right)^{1/\alpha}\right]\leqslant$$

$$\frac{a^\alpha+b^\alpha}{2(\ln b-\ln a)}\int_a^b\frac{f^\beta(x)}{x(a^\alpha+b^\alpha-x^\alpha)}\mathrm{d}x\leqslant$$

$$\min\{f^\beta(a)+\lambda_0 f^\beta(b),f^\beta(b)+\lambda_0 f^\beta(a)\}\cdot$$

$$\frac{a^\alpha+b^\alpha}{4(\ln b-\ln a)}\int_a^b\frac{1}{x(a^\alpha+b^\alpha-x^\alpha)}\cdot$$

$$\left[h\left(\frac{b^\alpha-x^\alpha}{b^\alpha-a^\alpha}\right)+h\left(\frac{x^\alpha-a^\alpha}{b^\alpha-a^\alpha}\right)\right]\mathrm{d}x.$$

证明 在定理 3 中取 $g(x)=x^\alpha(a^\alpha+b^\alpha-x^\alpha)$ 即可得证.

定理 4 设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数, $\beta>0,a,b>0,(2-\lambda_0)a^\alpha\neq\lambda_0^2 b^\alpha,(2-\lambda_0^2)b^\alpha\neq\lambda_0 a^\alpha,\int_a^{\lambda_0^{1/\alpha}\xi}x^{\alpha-1}f^\beta(x)\mathrm{d}x<+\infty$, $\int_{\lambda_0^{1/\alpha}\xi}^b x^{\alpha-1}f^\beta(x)\mathrm{d}x<+\infty,h\in L(0,1)$, 则有

$$\frac{1}{h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)}f^\beta\left\{\left[\frac{\lambda_0(2+\lambda_0+\lambda_0^2)}{2(1+\lambda_0)}\right]^{1/\alpha}\xi\right\}\leqslant$$

$$2\alpha h(1/2)\cdot\left[\frac{1}{\lambda_0^2 b^\alpha-(2-\lambda_0)a^\alpha}\int_a^{\lambda_0^{1/\alpha}\xi}x^{\alpha-1}f^\beta(x)\mathrm{d}x+\right.$$

$$\left.\frac{\lambda_0}{(2-\lambda_0^2)b^\alpha-\lambda_0 a^\alpha}\int_{\lambda_0^{1/\alpha}\xi}^b x^{\alpha-1}f^\beta(x)\mathrm{d}x\right]\leqslant$$

$$h(1/2)[1+(\lambda_0+\lambda_0^2)h(1/2)]\cdot$$

$$[f^\beta(a)+\lambda_0 f^\beta(b)]\int_0^1 h(u)\mathrm{d}u,\qquad(19)$$

其中 $\xi=\left(\frac{a^\alpha+\lambda_0 b^\alpha}{2}\right)^{1/\alpha}$.

证明 设 $I_1=\int_0^1 f^\beta((ua^\alpha+\lambda_0(1-u)\xi^\alpha)^{1/\alpha})\mathrm{d}u$, $I_2=\int_0^1 f^\beta((ub^\alpha+\lambda_0(1-u)\xi^\alpha)^{1/\alpha})\mathrm{d}u$, 则有

$$h(1/2)(I_1+\lambda_0 I_2)=$$

$$\frac{1}{h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)}\cdot$$

$$\left[h\left(\frac{\lambda_0}{1+\lambda_0}\right)h\left(\frac{1}{2}\right)(I_1+\lambda_0 I_2)+\right.$$

$$\left.\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)h\left(\frac{1}{2}\right)(I_1+\lambda_0 I_2)\right].\qquad(20)$$

对任意 $u\in[0,1]$, 由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义有

$$h(1/2)[f^\beta((ua^\alpha+\lambda_0(1-u)\xi^\alpha)^{1/\alpha}))+$$

$$\lambda_0[f^\beta((ub^\alpha+\lambda_0(1-u)\xi^\alpha)^{1/\alpha}))]\geqslant$$

$$f^\beta\left\{\left[\frac{ua^\alpha+\lambda_0(1-u)\xi^\alpha+\lambda_0(ub^\alpha+\lambda_0(1-u)\xi^\alpha)}{2}\right]^{1/\alpha}\right\}=$$

$$f^\beta\left\{\left[\frac{\lambda_0+\lambda_0^2+(1-\lambda_0)(2+\lambda_0)u}{2}\right]^{1/\alpha}\xi\right\},\qquad(21)$$

用 $1-u$ 替换式(21)中的 u, 得

$$h(1/2)[f^\beta(((1-u)a^\alpha+\lambda_0 u\xi^\alpha)^{1/\alpha}))+$$

$$\lambda_0[f^\beta(((1-u)b^\alpha+\lambda_0 u\xi^\alpha)^{1/\alpha}))]\geqslant$$

$$f^\beta\left\{\left[\frac{2-(1-\lambda_0)(2+\lambda_0)u}{2}\right]^{1/\alpha}\xi\right\},\qquad(22)$$

再由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义有

$$h\left(\frac{\lambda_0}{1+\lambda_0}\right)f^\beta\left\{\left[\frac{\lambda_0+\lambda_0^2+(1-\lambda_0)(2+\lambda_0)u}{2}\right]^{1/\alpha}\xi\right\}+$$

$$\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)f^\beta\left\{\left[\frac{2-(1-\lambda_0)(2+\lambda_0)u}{2}\right]^{1/\alpha}\xi\right\}\geqslant$$

$$f^\beta\left\{\left[\frac{\lambda_0}{1+\lambda_0}\cdot\frac{\lambda_0+\lambda_0^2+(1-\lambda_0)(2+\lambda_0)u}{2}+\right.\right.$$

$$\left.\left.\frac{\lambda_0}{1+\lambda_0}\cdot\frac{2-(1-\lambda_0)(2+\lambda_0)u}{2}\right]^{1/\alpha}\xi\right\}=$$

$$f^\beta\left\{\left[\frac{\lambda_0(2+\lambda_0+\lambda_0^2)}{2(1+\lambda_0)}\right]^{1/\alpha}\xi\right\},\qquad(23)$$

在式(21 ~ 23)中, 对 u 在 $[0,1]$ 上积分, 并与式(20)综合, 则式(19)的左边部分得证.

由 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义有

$$f^\beta((ua^\alpha+\lambda_0(1-u)\xi^\alpha)^{1/\alpha})\leqslant$$

$$h(u)f^\beta(a)+\lambda_0 h(1-u)f^\beta(\xi)\leqslant$$

$$h(u)f^\beta(a)+\lambda_0 h(1-u)h(1/2)\cdot$$

$$[f^\beta(a)+\lambda_0 f^\beta(b)],\qquad(24)$$

$$f^{\beta}((ub^{\alpha}+\lambda_0(1-u)\xi^{\alpha})^{1/\alpha}) \leqslant h(u)f^{\beta}(b)+\lambda_0 h(1-u)f^{\beta}(\xi) \leqslant h(u)f^{\beta}(b)+\lambda_0 h(1-u)h(1/2)\cdot[f^{\beta}(a)+\lambda_0 f^{\beta}(b)], \tag{25}$$

在式(24)~(25)中对 u 在$[0,1]$上积分,得

$$I_1 \leqslant \{[1+\lambda_0 h(1/2)]f^{\beta}(a)+\lambda_0^2 h(1/2)f^{\beta}(b)\}\int_0^1 h(u)\,\mathrm{d}u, \tag{26}$$

$$I_2 \leqslant \{\lambda_0 h(1/2)f^{\beta}(a)+[1+\lambda_0^2 h(1/2)]f^{\beta}(b)\}\int_0^1 h(u)\,\mathrm{d}u, \tag{27}$$

利用式(26~27)可证式(19)的右边成立.

用定理 4 的证明方法还可证明下面的定理 5.

定理 5　设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数,$\beta>0,a,b>0,(1-2\lambda_0)a^{\alpha}\neq\lambda_0 b^{\alpha},\lambda_0 b^{\alpha}\neq a^{\alpha},\int_{\lambda_0^{1/\alpha}a}^{\xi}x^{\alpha-1}f^{\beta}(x)\,\mathrm{d}x<+\infty,\int_{\xi}^{\lambda_0^{1/\alpha}b}x^{\alpha-1}\cdot f^{\beta}(x)\,\mathrm{d}x<+\infty$,则有

$$\frac{1}{h\left(\frac{\lambda_0}{1+\lambda_0}\right)+\lambda_0 h\left(\frac{1}{1+\lambda_0}\right)}f^{\beta}\left\{\left[\frac{\lambda_0(1+3\lambda_0)}{2(1+\lambda_0)}\right]^{1/\alpha}\xi\right\}\leqslant 2\alpha h\left(\frac{1}{2}\right)\left[\frac{1}{\lambda_0 b^{\alpha}+(1-2\lambda_0)a^{\alpha}}\int_{\lambda_0^{1/\alpha}a}^{\xi}x^{\alpha-1}f^{\beta}(x)\,\mathrm{d}x+\frac{\lambda_0}{\lambda_0 b^{\alpha}-a^{\alpha}}\int_{\xi}^{\lambda_0^{1/\alpha}b}x^{\alpha-1}f^{\beta}(x)\,\mathrm{d}x\right]\leqslant h(1/2)[\lambda_0+(1+\lambda_0)h(1/2)]\cdot[f^{\beta}(a)+\lambda_0 f^{\beta}(b)]\int_0^1 h(u)\,\mathrm{d}u,$$

其中 $\xi=\left(\frac{a^{\alpha}+\lambda_0 b^{\alpha}}{2}\right)^{1/\alpha}$.

注 1　在定理 1~5 及推论 1~3 中,将条件"$\beta>0$"改为"$\beta<0$",将结论中的"$\leqslant$"改为"$\geqslant$","min"改为"max",则相应的定理和推论仍然成立.

定理 6　设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数,$\beta>1,a,b>0,a\neq b,\int_a^b x^{\alpha-1}f(x)\,\mathrm{d}x<+\infty,h\in L(0,1)$,则有

$$\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b x^{\alpha-1}f(x)\,\mathrm{d}x\leqslant \min\{[f^{\beta}(a)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}b)]^{1/\beta},[f^{\beta}(b)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}a)]^{1/\beta}\}\left(\int_0^1 h(t)\,\mathrm{d}t\right)^{1/\beta}. \tag{28}$$

证明　利用 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数的定义,对任意 $\lambda\in[0,1]$ 有

$$f((\lambda a^{\alpha}+(1-\lambda)b^{\alpha})^{1/\alpha}) \leqslant \min\{[h(\lambda)f^{\beta}(a)+\lambda_0 h(1-\lambda)f^{\beta}(\lambda_0^{-1/\alpha}b)]^{1/\beta},[h(1-\lambda)f^{\beta}(b)+\lambda_0 h(\lambda)f^{\beta}(\lambda_0^{-1/\alpha}a)]^{1/\beta}\}, \tag{29}$$

在式(29)中对 λ 在$[0,1]$上积分并使用 Jensen 积分不等式(因 $\beta>1$,故 $x^{1/\beta}$ 为凹函数),得

$$\int_0^1 f((\lambda a^{\alpha}+(1-\lambda)b^{\alpha})^{1/\alpha})\,\mathrm{d}\lambda \leqslant \min\left\{\int_0^1[h(\lambda)f^{\beta}(a)+\lambda_0 h(1-\lambda)f^{\beta}(\lambda_0^{-1/\alpha}b)]^{1/\beta}\mathrm{d}\lambda,\int_0^1[h(1-\lambda)f^{\beta}(b)+\lambda_0 h(\lambda)f^{\beta}(\lambda_0^{-1/\alpha}a)]^{1/\beta}\mathrm{d}\lambda\right\}\leqslant \min\left\{\left[\int_0^1(h(\lambda)f^{\beta}(a)+\lambda_0 h(1-\lambda)f^{\beta}(\lambda_0^{-1/\alpha}b))\mathrm{d}\lambda\right]^{1/\beta},\left[\int_0^1(h(1-\lambda)f^{\beta}(b)+\lambda_0 h(\lambda)f^{\beta}(\lambda_0^{-1/\alpha}a))\mathrm{d}\lambda\right]^{1/\beta}\right\}=\min\left\{[f^{\beta}(a)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}b)]^{1/\beta},[f^{\beta}(b)+\lambda_0 f^{\beta}(\lambda_0^{-1/\alpha}a)]^{1/\beta}\right\}\left(\int_0^1 h(t)\,\mathrm{d}t\right)^{1/\beta}, \tag{30}$$

其中对式(30)左边积分利用积分变量代换得

$$\int_0^1 f((\lambda a^{\alpha}+(1-\lambda)b^{\alpha})^{1/\alpha})\,\mathrm{d}\lambda=\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_a^b x^{\alpha-1}f(x)\,\mathrm{d}x,$$

故式(28)得证.

定理 7　设 $f:(0,+\infty)\to(0,+\infty)$ 是 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数,$\beta<0,h(\lambda)$ 分别在点 $\lambda=0$ 和 $\lambda=1$ 处右可微和左可微,且 $h(0)=0,h(1)=1,h'_+(0)>0,0<a<b,y=\left(\frac{\lambda_0}{b-a}\int_a^b x^{\alpha}\mathrm{d}x\right)^{1/\alpha}$,即当 $\alpha\neq-1$ 时,$y=\left(\frac{\lambda_0(b^{\alpha+1}-a^{\alpha+1})}{(\alpha+1)(b-a)}\right)^{1/\alpha}$,当 $\alpha=-1$ 时,$y=\frac{b-a}{\lambda_0(\ln b-\ln a)}$,若 f 在点 y 处可微,$f\in L[a,b]$,则有

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\geqslant\left(\frac{h'_-(1)}{\lambda_0 h'_+(0)}\right)^{1/\beta}f(y). \tag{31}$$

证明　注意到 $\beta<0$,由引理 1,对任意 $x>0$,有

$$0<\lambda_0 h'_+(0)f^{\beta}(x)\leqslant h'_-(1)f^{\beta}(y)+\frac{\beta y^{1-\alpha}f^{\beta-1}(y)f'(y)}{\alpha}(\lambda_0 x^{\alpha}-y^{\alpha}), \tag{32}$$

由式(32)得

$$f(x) \geqslant f(y)\left[\frac{h'_-(1)}{\lambda_0 h'_+(0)} + \frac{\beta y^{1-\alpha} f'(y)}{\lambda_0 h'_+(0) f(y)}(\lambda_0 x^\alpha - y^\alpha)\right]^{1/\beta}, \tag{33}$$

在式(33)中对 x 在$[a,b]$上积分并使用 Jensen 积分不等式(因 $\beta < 0$, 故 $x^{1/\beta}$ 为凸函数),得

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geqslant \frac{f(y)}{b-a}\int_a^b\left[\frac{h'_-(1)}{\lambda_0 h'_+(0)} + \frac{\beta y^{1-\alpha} f'(y)}{\lambda_0 h'_+(0) f(y)}(\lambda_0 x^\alpha - y^\alpha)\right]^{1/\beta}\mathrm{d}x \geqslant f(y)\left\{\frac{1}{b-a}\int_a^b\left[\frac{h'_-(1)}{\lambda_0 h'_+(0)} + \frac{\beta y^{1-\alpha} f'(y)}{\lambda_0 h'_+(0) f(y)}(\lambda_0 x^\alpha - y^\alpha)\right]\mathrm{d}x\right\}^{1/\beta} =$$

$$f(y)\left\{\frac{h'_-(1)}{\lambda_0 h'_+(0)} + \frac{\beta y^{1-\alpha} f'(y)}{\lambda_0 h'_+(0) f(y)}\cdot\left[\frac{\lambda_0}{b-a}\int_a^b x^\alpha \mathrm{d}x - y^\alpha\right]\right\}^{1/\beta} = \left(\frac{h'_-(1)}{\lambda_0 h'_+(0)}\right)^{1/\beta} f(y),$$

故式(31)得证.

定理 8 设 $f:(0,+\infty)\to(0,+\infty)$ 是可微的 $(\alpha,\beta,\lambda,\lambda_0,h)$ 凸函数, $\lambda_0 \in (0,1)$, $0 < a < b$, $h(0) = 0$, $h(1) = 1$, f' 在$[a,b]$上连续,则有

$$\left(\frac{1}{\alpha} + \frac{k}{\beta}\right)\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{bf(b) - af(a)}{\alpha}, \tag{34}$$

$\beta > 0(\beta < 0)$ 时,有

$$\left(k + \frac{1}{\alpha}\right)\int_a^b f^\beta(x)\,\mathrm{d}x \leqslant (\geqslant)\ \frac{bf^\beta(b) - af^\beta(a)}{\alpha}, \tag{35}$$

其中 $k = \dfrac{h'(1) - \lambda_0 h'(0)}{1 - \lambda_0}$, 式(34~35)等号成立的充要条件是 $f(x) = x^{k\alpha/\beta}$.

证明 在引理 1 的式(2)中取 $x = y$ 并整理得

$$\frac{1}{\alpha}xf'(x) \geqslant \frac{k}{\beta}f(x), \tag{36}$$

在式(36)中对 x 在$[a,b]$上积分并整理,则式(34)得证. 将式(36)乘以 $f^{\beta-1}(x)$ 得

$$\frac{1}{\alpha}xf^{\beta-1}(x)f'(x) \geqslant \frac{k}{\beta}f^\beta(x), \tag{37}$$

在式(37)中对 x 在$[a,b]$上积分并整理,则式(35)得证.

式(34)~(35)等号成立的充要条件是式(36)的等号成立,即 $\dfrac{1}{\alpha}xf'(x) = \dfrac{k}{\beta}f(x)$, 由此解得 $f(x) = x^{k\alpha/\beta}$.

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Hermite-Hadamard Type Inequalities for $(\alpha, \beta, \lambda, \lambda_0, h)$- Convex Function

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Abstract: Staring from the definition of $(\alpha, \beta, \lambda, \lambda_0, h)$-convex function, Hermite-Hadamard type inequalities for $(\alpha, \beta, \lambda, \lambda_0, h)$-convex function is established by using mathematical analysis. In the case where h is differentiable function with $h(0)=0$, $h(1)=1$, Hermite-Hadamard type inequalities for $(\alpha, \beta, \lambda, \lambda_0, h)$-convex function is obtained by using the relationship between $(\alpha, \beta, \lambda, \lambda_0, h)$-convex function and its derivative.

Keywords: convex function; $(\alpha, \beta, \lambda, \lambda_0, h)$-convex function; Hermite-Hadamard type inequality