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依赖密度的 Boussinesq 方程组的局部强解的存在唯一性

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摘要: 本文主要研究了非齐次不可压 Boussinesq 方程组的局部强解的存在唯一性,并得到了强解的一致估计,且这些估计不依赖与初始密度 ρ_0 的下界.

关键词: 非齐次 Boussinesq 方程组; 强解; 存在唯一性

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1 引言及主要结论

Boussinesq 方程组是流体方程中一类重要的方程. 本文主要考虑如下非齐次不可压 Boussinesq 方程组的初边值问题:

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ \operatorname{div}u=0,\\ (\rho u)_t+\operatorname{div}(\rho u\otimes u)-\operatorname{div}(2\mu du)+\nabla p=\rho f+f\theta,\\ C_v(\partial_t(\rho\theta)+\operatorname{div}(\rho u\theta))-\operatorname{div}(k\nabla\theta)=2\mu|du|^2,\\ \rho|_{t=0}=\rho_0,\rho u|_{t=0}=\rho_0u_0,\rho\theta|_{t=0}=\rho_0\theta_0,\\ u|_{\partial\Omega}=0,\nabla\theta\cdot n|_{\partial\Omega}=0.\end{cases}\tag{1.1}$$

上面方程组中,$\Omega\subseteq R^3$ 是有界并且边界光滑的区域,研究的流体速度函数定义为 $u=u(x,t)$,温度函数、密度函数、压力函数定义为 $\theta=\theta(x,t)$. 定义在外力作用下的向量函数 $f=f(x,t)$,并且已知初始速度 $u=u_0(x)$,初始动量 $\rho_0u_0=\rho_0u_0(x)$. 这里 C_v,μ,κ 均为 θ 的正的函数,C_v、μ、κ 分别表示比热容、流体粘性系数、热传导系数.

近年来,有关 Boussinesq 方程组的研究有了新的进展. 1999 年,Kageir 证明了三维粘性 Boussinesq 方程组弱解的整体存在性[1]. 另外,李现今[2] 在 2010 年又证明了三维 Boussinesq 方程组大解的全局 L^2 稳定性.

本文研究依赖 ρ 的三维 Boussinesq 方程组强解的局部存在唯一性.

定理 1.1 设 $\Omega\subseteq\mathbf{R}^3$ 为边界光滑的有界开区域,T 是一个正常数,假设 (ρ_0,u_0,θ_0,f) 满足正则条件:

$$\rho_0\geqslant 0,\rho_0\in H^{-1,q}(\Omega),\theta_0\in H^2(\Omega),$$
$$\nabla\theta_0\cdot n|_{\partial\Omega}=0,u_0\in H^2(\Omega)\cap H_0^1(\Omega),$$
$$\operatorname{div}u_0=0,f\in H^1(\Omega\times(0,T)),$$
$$f_t\in L^2(0,T;L^2(\Omega)),$$

若 (ρ,u,p,θ) 满足下面的条件:

$$\begin{cases}-\operatorname{div}(2\mu_0du_0)+\nabla p_0-\rho_0f-f\theta_0=\rho_0^{\frac{1}{2}}g_1,\\ \operatorname{div}(k_0\nabla\theta_0)-2\mu_0|du_0|^2=\rho_0^{\frac{1}{2}}g_2.\end{cases}\tag{1.2}$$

这里 $(g_1,g_2)\in L^2(\Omega)$,$P_0\in H^1(\Omega)$ 而且系数满足:

$$0<\mu,C_V,k\in C^1(R^2),\mu=\mu(\rho,\rho\theta),$$
$$C_V=C_V^1(\rho,\rho\theta),k=k(\rho,\rho\theta).$$

那么初边值问题(1.1)存在唯一的局部强解 (ρ,u,p,θ) 满足:

$$\rho\in C([0,T_*]:H^{-1,q}),\rho_t\in C([0,T_*]:L^q),(u,\theta)\in C([0,T_*]:H^2)\cap L^2(0,T_*:W^{2,q})P\in C([0,T_*]:H^1)\cap L^2(0,T_*:H^{-1,q}),(u_t,\theta_t)\in L^2(0,T_*:H^1),$$
$$(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\in L^\infty(0,T_*:L^2).\tag{1.3}$$

其中 $T_*>0$ 是某常数.

2 解的一致估计

为了证明定理 1.1,考虑线性问题:

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$$\begin{cases}\rho_t + \operatorname{div}(\rho v) = 0,\\ \operatorname{div} u = 0,\\ (\rho u)_t + \operatorname{div}(\rho v \otimes u) - \operatorname{div}(2\mu_0 du) + \nabla p\\ = \operatorname{div}(2(\mu - \mu_0) dv) + \rho f + f\theta,\\ C_V(\partial_t(\rho\theta) + \operatorname{div}(\rho v\theta)) - \operatorname{div}(k_0 \nabla \theta)\\ = \operatorname{div}((k - k_0) \nabla \eta) + 2\mu |dv|^2,\\ \rho|_{t=0} = \rho_0(x), \theta|_{t=0} = \theta_0(x), u|_{t=0} = u_0(x),\\ u|_{\partial\Omega} = 0, \nabla\theta \cdot n|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

这里记:

$$\mu := \mu(\rho, \rho\eta), C_v := C_V(\rho, \rho\eta), k := k(\rho, \rho\eta),$$
$$\mu_0 := \mu(\rho_0, \rho_0\theta_0), k_0 := k(\rho_0, \rho_0\theta_0)$$

假定初值 (ρ_0, u_0, θ_0) 满足:

$$\rho_0 \geqslant 0, \rho_0 \in W^{1,q}(\Omega), \theta_0 \in H^2(\Omega),$$
$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \operatorname{div} u_0 = 0,$$
$$\begin{cases} -\operatorname{div}(2\mu_0 du_0) + \nabla p_0 - \rho_0 f - f\theta_0 = \rho_0^{\frac{1}{2}} g_1,\\ \operatorname{div}(k_0 \nabla \theta_0) - 2\mu_0 |du_0|^2 = \rho_0^{\frac{1}{2}} g_2. \end{cases} \tag{2.2}$$

$q \in [3,6]$, $(g_1, g_2) \in L^2(\Omega)$, $p_0 \in H^1(\Omega)$, $f \in H^1(\Omega \times (0,T))$, $f_t \in L^2(0,T; L^2(\Omega))$. 假定已知向量对 (v,η) 满足:

$$(v,\eta) \in C([0,T]; H^2(\Omega)) \cap L^2(0,T; W^{2,q}(\Omega)),$$
$$(v_t, \eta_t) \in L^2(0,T; H^1(\Omega)). \tag{2.3}$$
$$\operatorname{div} v = 0, v|_{\partial\Omega} = 0, \nabla\eta \cdot n|_{\partial\Omega} = 0$$

引理 2.1

$$\left|\sqrt{\rho} u_t(t)\right|_{L^2}^2 + |u(t)|_{H_0^1}^2 + \int_0^t |u_t(s)|_{H_0^1}^2 ds \leqslant M(c_0). \tag{2.11}$$

$$|(\nabla u(t), p(t))|_{H^1} \leqslant M(c_0) c_1^2 c_2^{\frac{3}{q_1}}. \tag{2.12}$$

$$\int_0^t |u(s)|_{W^{2,q}}^2 + |p(s)|_{W^{1,q}}^2 ds \leqslant M(c_0). \tag{2.13}$$

这里 $0 \leqslant t \leqslant T$, C 为一个正常数, 而且 $M := M(\cdot)$ 定义为一个一般的从 $[1, +\infty)$ 到 $[1, +\infty)$ 的连续增函数, 它仅仅依赖于参数 C. 选择固定的 $c_1 := M(c_0)$, $c_2 := M(c_0) c_1^2 c_2^{\frac{3}{q_1}}$,

可以得到结论:

$$\sup_{0\leqslant t\leqslant T_2} u(t)_{H_0^1} + \int_0^{T_2} u_t(t)_{H_0^1}^2 + u(t)_{W^{2,q}}^2 dt \leqslant c_1. \tag{2.14}$$

$$\sup_{0\leqslant t\leqslant T_2} (u,\theta)(t)_{H^2} + \int_0^{T_2} \nabla\theta_t(t)_{L^2}^2 + \theta(t)_{W^{2,q}}^2 dt \leqslant c_2. \tag{2.15}$$

$$\sup_{0\leqslant t\leqslant T_2} \theta(t) - \theta(0)_{W^{1,4}} \leqslant c_2^{-1}, \int_0^{T_2} \sqrt{\rho_0}\theta_t(t)_{L^2}^2 dt \leqslant c_2^{-6}. \tag{2.16}$$

$$\sup_{0\leqslant t\leqslant T_2} (\rho(t)_{W^{1,q}} + \rho_t(t)_{L^q} + p(t)_{H^1}) + \sup_{0\leqslant t\leqslant T_2} (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t)_{L^2} + \int_0^{T_2} p(t)_{W^{1,q}}^2 dt \leqslant c_2. \tag{2.17}$$

定理 2.1　假定 (v,η) 满足(2.3), 用 T_* 代替 T 且 $(v(0), \eta(0)) = (u_0, \theta_0)$,

$$\sup_{0\leqslant t\leqslant T_*} v(t)_{H^1} + \int_0^{T_*} v_t(t)_{H^1}^2 + v(t)_{W^{2,q}}^2 dt \leqslant c_1,$$
$$\sup_{0\leqslant t\leqslant T_*} (v,\eta)(t)_{H^2} + \int_0^{T_*} \nabla\eta_t{}_{L^2}^2 + \eta(t)_{W^{2,q}}^2 dt \leqslant c_2,$$
$$\sup |\eta(t) - \eta(0)|_{W^{1,1}} \leqslant c_2^{-1},$$
$$\int_0^{T_*} \left|\sqrt{\rho_0}\eta_t(t)\right|_{L^2}^2 dt \leqslant c_2^{-6}. \tag{2.18}$$

那么线性问题(2.1)在 $[0,T_*]$ 存在唯一的强解 (ρ,u,p,θ) 满足估计(2.14)-(2.17), 而且有下面的正则性:

$\rho \in C([0,T_*]; W^{1,q})$, $\rho_t \in C([0,T_*]; L^q)$, $(u,\theta) \in C([0,T_*]; H^2) \cap L^2(0,T_*; W^{2,q})$,

$p \in C([0,T_*]; H^1) \cap L^2(0,T_*; H^{1,q})$,

$(u_t, \theta_t) \in L^2(0,T_*; H^1)$, $(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t) \in L^\infty(0,T_*; L^2)$

因为证明过程和参考文献[3-4]类似, 在此省略.

3　定理 1.1 的证明

证明是建立在定理 2.1 的结果上, 设 $u^0, \theta^0 \in C([0, +\infty); H^2) \cap L^2(0, +\infty; H^3)$ 分别满足下面的线性抛物方程:

$$\partial_t u^0 - \Delta u^0 + \nabla p^0 = 0, \operatorname{div} u^0 = 0, x \in \Omega,$$
$$u_0|_{\partial\Omega} = 0, u^0|_{t=0} = u_0, \partial_t\theta^0 - \Delta\theta^0 = 0, x \in \Omega,$$
$$\nabla\theta^0 \cdot n|_{\partial\Omega} = 0, \theta^0|_{t=0} = \theta_0.$$

根据抛物方程的性质, 得到:

$$\sup_{0\leqslant t\leqslant 1} (u^0,\theta^0)(t)_{H^2} + \int_0^1 (u_t^0, \theta_t^0)(t)_{H^1}^2 + (u^0,\theta^0)(t)_{W^{2,q}}^2 dt \leqslant c_1,$$

而且, 由于 $\theta^0 \in C([0, +\infty); H^2)$, $\theta_t^0 \in L^2(0, +\infty; H^1)$ 和 $\rho_0 \in L^3$, 所以存在一个小的时间 $T_* \in (0,1)$ 且满足:

$$\sup_{0\leqslant t\leqslant T_*} |\theta^0(t) - \theta_0|_{W^{1,4}} \leqslant c_2^{-1},$$
$$\int_0^{T_*} \left|\sqrt{\rho_0}\theta_t^0(t)\right|_{L^2}^2 dt \leqslant c_2^{-6}. \tag{3.2}$$

根据(3.1)和(3.2), 如果 $(v,\eta) := (u^0, \theta^0)$ 满足定

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理(2.1)的假设,那么线性问题(2.1)在$(0,T_*)$内也存在唯一的强解$(\rho,u,p,\theta)=(\rho^1,u^1,p^1,\theta^1)$且$(v,\eta)=(u^0,\theta^0)$,满足估计(2.14)-(2.17),若$(v,\eta)=(u^1,\theta^1)$也满足定理(2.1)的假设,那么线性问题(2.1)在$(0,T_*)$内存在唯一的强解$(\rho,u,p,\theta)=(\rho^2,u^2,p^2,\theta^2)$且$(v,\eta)=(u^1,\theta^1)$,满足估计(2.14)-(2.17). 根据这种迭代过程,能够定义一个序列$(\rho^i,u^i,p^i,\theta^i)$,对任意的$i\geqslant 1$,$(\rho,u,p,\theta)=(\rho^i,u^i,p^i,\theta^i)$是线性问题(2.1)在$(0,T_*)$内存在的唯一解,且$(v,\eta)=(u^{i-1},\theta^{i-1})$并且满足一致估计:

$$\sup_{0\leqslant t\leqslant T_*}\left(\rho^i(t)_{W^{1,q}}+\rho_t^i(t)_{L^q}\right)+\sup_{0\leqslant t\leqslant T_*}\left((u^i,\theta^i)(t)_{H^2}+p^i(t)_{H^1}\right)+\sup_{0\leqslant t\leqslant T_*}\left(\int_0^{T_*}(u_t^i,\theta_t^i)(t)_{H^1}^2+(u^i,\theta^i)(t)_{W^{2,q}}^2+p^i(t)_{W^{1,q}}^2dt+(\sqrt{\rho^i}u_t^i,\sqrt{\rho^i}\theta_t^i)(t)_{L^2}\right)\leqslant\hat{c}. \tag{3.3}$$

上面的$\hat{c}$是一个正常数,它仅依赖于c_0和参数C,不依赖i. 能够证明这个全序列$(\rho^i,u^i,p^i,\theta^i)_{i\geqslant 1}$强收敛于原非线性问题(1.1)的解. 从而能够断定(ρ,u,θ)是初始问题(1.1)的弱解,根据(2.14)-(2.17),有:

$$\begin{gathered}\rho\in L^\infty(0,T_*;W^{1,q}),\rho_t\in L^\infty(0,T_*;L^q),\\(u,\theta)\in L^\infty(0,T_*;H^2)\cap L^2(0,T_*;W^{2,q}),\\p\in L^\infty(0,T_*;H^1)\cap L^2(0,T_*;W^{1,q}),\\(u_t,\theta_t)\in L^2(0,T_*;H^1),\\(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\in L^\infty(0,T_*;L^2).\end{gathered} \tag{3.4}$$

注:根据参考文献[3]易证明弱解(ρ,u,p,θ)关于时间的连续性和解的唯一性. 在此省略.

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Existence and Uniqueness of Local Strong Solutions for Boussinesq Systems of Density Dependent Equations

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Abstract: This paper mainly studied the existence and uniqueness of the local strong solutions of the inhomogeneous incompressible Boussinesq equations and obtained the uniform estimation of the strong solutions, which were not dependent on the lower bounds of the initial density.

Keywords: nonhomogeneous Boussinesq equations; strong solutions; existence and uniqueness

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