带有 p-Laplacian 算子的奇异分数阶边值问题解的存在唯一性

王 丽

(曲阜师范大学数学科学学院,山东 曲阜 273165)

摘要:本文依据混合单调算子理论以及相关算子方程的不动点结果,研究带有 p-Laplacian 算子的 Caputo 型奇异分数阶边值问题解的存在唯一性.

关键词:奇异分数阶微分方程;p-Laplacian 算子;混合单调算子;积分边值条件

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0 引 言

本文研究下述分数阶微分方程边值问题

$$\begin{cases}(\varphi_p({}^cD_{0+}^{\alpha}u(t)))'+f(t,u(t),u(t))=0,t\in(0,1),\\ u(0)=u''(0)={}^cD_{0+}^{\alpha}u(0)=0,\\ u(1)=\lambda\int_0^1 u(s)\mathrm{d}s\end{cases}\tag{0.1}$$

解的存在唯一性.其中 $2<\alpha\leqslant 3$,$0<\lambda<2$,${}^cD_{0+}^{\alpha}$ 是 α 阶 Caputo 型分数阶导数,$\varphi_p(s)=|s|^{p-2}s$,$p>1$,$f:(0,1)\times[0,+\infty)\times(0,+\infty)\to[0,+\infty)$,$f(t,x,y)$ 在 $y=0$ 和 $t=0,1$ 可以是奇异的.众所周知,φ_p 是可逆的,它的逆算子是 φ_q,其中常数 $q>1$ 满足 $\frac{1}{p}+\frac{1}{q}=1$.

近年来,分数阶微分方程在数学及其应用中都引起了广泛的关注,特别是应用于工程和科学研究的许多领域,如物理、化学、电力科学和生物学等,详情参见文献[1-7].带有 p-Laplacian 算子的分数阶微分方程的理论背景和应用,详情参见文献[8-10].此外,在该类分数阶微分方程中,其边值条件是积分形式的情况也引起了许多的关注,详情参见文献[11-14].

文献[10]研究了下列带有 p-Laplacian 算子的分数阶微分方程边值问题.

$$\begin{cases}(\varphi_p({}^cD_{0+}^{\alpha}u(t)))'+f(t,u(t))=0,\\ u(0)=D_{0+}^{\alpha}u(0)=0,\\ {}^cD_{0+}^{\beta}u(0)={}^cD_{0+}^{\beta}u(1)=0\end{cases}$$

解的存在唯一性.其中 $0<\beta\leqslant 1$,$2<\alpha<2+\beta$,D_{0+}^{α} 和 ${}^cD_{0+}^{\beta}$ 分别是 α 阶 Riemann-Liouville 型分数阶导数和 β 阶 Caputo 型分数阶导数,$f:[0,1]\times\boldsymbol{R}\to\boldsymbol{R}$ 是连续函数.应用 Guo-Krasnoselskii 不动点定理和 Banach 压缩映射原理证明了解的存在唯一性.

文献[14]研究了如下带有积分边值条件的非线性分数阶微分方程

$$\begin{cases}({}^cD_{0+}^{\alpha}u(t))'+f(t,u(t))=0,t\in[0,1],\\ u(0)=u''(0)=0,\\ u(1)=\lambda\int_0^1 u(s)\mathrm{d}s\end{cases}$$

正解的存在性.其中 $2<\alpha<3$,$0<\lambda<2$,$f:[0,1]\times[0,+\infty)\to[0,+\infty)$ 是连续函数.运用 Guo-Krasnoselskii 不动点定理证明了正解的存在性.

受以上文献的启发,研究问题(0.1)解的存在性和唯一性,其证明的主要方法是运用混合单调算子的不动点定理.不同于以上其他文章,本文中的非线性项 f 包含了可奇异的点.

1 预备知识

下面介绍分数阶微积分理论的相关定义和引理,其详情参见文献[1-3].

定义 1.1 函数 $u:[0,+\infty)\to\boldsymbol{R}$ 的 $\alpha>0$ 阶

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Riemann-Liouville 型分数阶积分为

$$I_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s,$$

上式右端在 $\boldsymbol{R}^+$ 内是诸点定义的. 其中 $\Gamma(\alpha)$ 是欧拉 Gamma 函数,定义如下

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}\mathrm{d}t, \alpha > 0.$$

定义 1.2 函数 $u:[0,+\infty)\to\boldsymbol{R}$ 的 $\alpha>0$ 阶 Caputo 型分数阶导数为

$${}^{c}D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)\mathrm{d}s,$$

其中 $n=[\alpha]+1$.

引理 1.1[1] 设 $\alpha>0$, 则分数阶微分方程

$${}^{c}D^{\alpha}u(t) = 0$$

有唯一解,其表达式为

$$u(t) = \sum_{k=0}^{[\alpha]}\frac{u^{(k)}(0)}{k!}t^k.$$

引理 1.2[1] 设 $\alpha>0$ 及 $u\in C[0,1]$, 则 ${}^{c}D_{0+}^{\alpha}I_{0+}^{\alpha}u(t) = u(t)$.

引理 1.3[1] 若 $\alpha>0$, $n=[\alpha]+1$ 及 $u\in AC^n[0,1]$, 则

$$I_{0+}^{\alpha}{}^{c}D_{0+}^{\alpha}u(t) = u(t) + C_1 + C_2t + \cdots + C_nt^{n-1}.$$

设 $(E,\|\cdot\|)$ 是实 Banach 空间, P 是 E 中的锥, 且 θ 是 E 中的零元素, P 在 E 中诱导出偏序“$\leqslant$”. 如果存在常数 $C>0$, 满足对任意 $x,y\in E$ 及 $\theta\leqslant x\leqslant y$, 都有 $\|x\|\leqslant C\|y\|$, 则锥 P 称为正规的. 对于任意 $x,y\in E$, 当存在常数 $\lambda,\mu>0$, 满足 $\lambda y\leqslant x\leqslant\mu y$ 时,用 $x\sim y$ 表示. 显然, $\sim$ 是等价关系. 给定 $h>\theta$, 用 P_h 表示集合 $P_h=\{x\in E: x\sim h\}$.

定义 1.3[3] 定义算子 $A:P\times P\to P$, 若 $A(x,y)$ 关于 x 递增且关于 y 递减,即当 $u_i,v_i(i=1,2)\in P$, $u_1\leqslant u_2$, $v_1\geqslant v_2$ 时,有

$$A(u_1,v_1)\leqslant A(u_2,v_2),$$

则称 A 为混合单调算子. 若 $A(x,x)=x$. 则称元素 x 为 A 的一个不动点.

引理 1.4[3] 设 $P_h\subset P$, 若 $A:P_h\times P_h\to P_h$ 是混合单调算子,并且满足下列条件: 对于任意的 $u,v\in P_h$ 和 $t\in(0,1)$, 存在 $\varphi(t)\in(t,1)$, 有

$$A(tu,t^{-1}v)\geqslant\varphi(t)A(u,v),$$

则算子方程 $A(x,x)=x$ 有唯一解 $x^*\in P_h$. 同时,对于任意的初值 $x_0,y_0\in P_h$, 序列

$$x_n=A(x_{n-1},y_{n-1}),\ y_n=A(y_{n-1},x_{n-1}),\ n=1,2,\cdots,$$

当 $n\to\infty$ 时,有 $\|x_n-x^*\|\to0$, $\|y_n-x^*\|\to0$.

2 主要结果

引理 2.1 设 $y\in C[0,1]\cap L[0,1]$, $2<\alpha\leqslant3$, $0<\lambda<2$. 则下列边值问题

$$\begin{cases}(\varphi_p({}^{c}D_{0+}^{\alpha}u(t)))' + y(t) = 0, 0<t<1,\\ u(0) = u''(0) = {}^{c}D_{0+}^{\alpha}u(0) = 0,\\ u(1) = \lambda\int_0^1 u(s)\mathrm{d}s\end{cases} \tag{2.1}$$

有唯一解

$$u(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s y(\tau)\mathrm{d}\tau\Big)\mathrm{d}s, \tag{2.2}$$

其中

$$G(t,s) = \frac{1}{(2-\lambda)\Gamma(\alpha+1)}\begin{cases}2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (2-\lambda)\alpha(t-s)^{\alpha-1}, & 0\leqslant s\leqslant t\leqslant1,\\ 2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s), & 0\leqslant s\leqslant t\leqslant1.\end{cases} \tag{2.3}$$

证明 对(2.1)的第一个方程在 $[0,t]$ 上积分,得

$$\varphi_p({}^{c}D_{0+}^{\alpha}u(t)) - \varphi_p({}^{c}D_{0+}^{\alpha}u(0)) = -\int_0^t y(s)\mathrm{d}s,$$

将 ${}^{c}D_{0+}^{\alpha}u(0)=0$ 带入上式,有

$$\varphi_p({}^{c}D_{0+}^{\alpha}u(t)) = -\int_0^t y(s)\mathrm{d}s,$$

对上式两边作用 φ_q, 得

$${}^{c}D_{0+}^{\alpha}u(t) = -\varphi_q\Big(\int_0^t y(s)\mathrm{d}s\Big),$$

根据引理 1.3,对上式两边求 α 阶积分,有

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\Big(\int_0^s y(\tau)\mathrm{d}\tau\Big)\mathrm{d}s + C_0 + C_1t + C_2t^2. \tag{2.4}$$

再把 $u(0)=u''(0)=0$ 带入(2.4),得 $C_0=C_2=0$, 因此

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\Big(\int_0^s y(\tau)\mathrm{d}\tau\Big)\mathrm{d}s + C_1t. \tag{2.5}$$

对(2.5)在 $[0,1]$ 上积分,即

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$$\int_0^1 u(t)\,\mathrm{d}t = -\frac{1}{\Gamma(\alpha)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\alpha}\varphi_q\Big(\int_0^s y(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s + \frac{1}{2}C_1. \tag{2.6}$$

由边值条件 $u(1) = \lambda\int_0^1 u(s)\,\mathrm{d}s$, 结合(2.5)和(2.6),得

$$C_1 = \frac{2}{(2-\lambda)\Gamma(\alpha+1)}\cdot \int_0^1 (1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)\,\varphi_q\Big(\int_0^s y(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s. \tag{2.7}$$

把(2.7)带入(2.5),可以得到边值问题(2.1)的唯一解是

$$u(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s y(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s. \tag{2.8}$$

其中 $G(t,s)$ 是(2.3)所示的格林函数. 证毕.

积分方程 $u(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s y(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s$ 是边值问题(2.1)的解当且仅当 $u(t) = \int_0^1 G(t,s)\cdot \varphi_q\Big(\int_0^s f(\tau,u(\tau),u(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s(u \in P_h)$ 是边值问题(0.1)的解,其中 $G(t,s)$ 是(2.3)所示的格林函数, $y \in C[0,1]\cap L[0,1]$, $f:(0,1)\times[0,+\infty)\times(0,+\infty)\to[0,+\infty)$ 是连续函数.

注 1　引理 2.1 中的边值问题与文献[14]中的边值问题方程不同,但是部分边值条件相同,使得所求的格林函数相同. 所以本文中的格林函数与文献[14]中的格林函数具有相同的性质.

引理 2.2[14]　由(2.3)定义的格林函数 $G(t,s)$ 满足下列性质:

(1)对于任意的 $s \in (0,1)$, $\lambda \in (0,2)$, 有 $G(1,s) > 0$;

(2)对于任意的 $t,s \in (0,1)$, $\lambda \in (0,2)$, 有 $G(t,s) > 0$;

(3)对于任意的 $t,s \in (0,1)$, $\lambda \in (0,2)$, 有 $J(s)t \leqslant G(t,s) \leqslant K(s)t$,

其中

$$J(s) = G(1,s),$$

$$K(s) = \frac{2}{(2-\lambda)\Gamma(\alpha+1)}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s).$$

设 Banach 空间 $E = C[0,1]$, E 中的范数是

$$\|x\| = \sup\{|x(t)|: t \in [0,1]\}.$$

定义 E 中的锥 P 为

$$P = \{u \in E: u(t) \geqslant 0, \text{对任意的 } t \in [0,1]\},$$

并定义

$$P_h = \left\{u \in P: a_u h(t) \leqslant u(t) \leqslant b_u h(t), \atop \text{对任意的 } t \in [0,1], \text{存在} a_u \leqslant b_u\right\}, \tag{2.9}$$

其中 $h(t) = t, t \in (0,1)$.

在空间 E 中有锥 P 导出的如下偏序关系

$x,y \in E, x \leqslant y \Leftrightarrow x(t) \leqslant y(t)$, 对任意的 $t \in [0,1]$.

给出下列假设:

(H_1) 对任意的 $t \in (0,1)$, $f(t,x,y)$ 关于 x 递增且关于 y 递减;

(H_2) 对任意的 $x \in [0,+\infty)$, $y \in (0,+\infty)$, $t \in (0,1)$, 存在 $\gamma \in (0,p-1)$, 当 $0 < r < 1$ 时,有

$$f(t,rx,r^{-1}y) \geqslant r^{\gamma}f(t,x,y);$$

(H_3) 对任意的 $t \in (0,1)$, 有 $f(t,0,1) \not\equiv 0$ 并且

$$\int_0^1 f(t,1,t)\,\mathrm{d}t < +\infty.$$

注 2　由 (H_2) 可以推出,当 $r \geqslant 1$ 时,有 $f(t,rx,r^{-1}y) \leqslant r^{\gamma}f(t,x,y)$;由 (H_1) 知 $f(t,0,1) < f(t,1,t)$, 再根据 (H_3) 中的 $f(t,0,1) \not\equiv 0$, 可知 $f(t,1,t) \not\equiv 0$.

定理 2.1　设 (H_1)—(H_3) 成立,则问题(0.1)有唯一正解 $u^* \in P_h$. 对于任意的初值 $u_0, v_0 \in P_h$, 序列

$$u_{n+1}(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u_n(\tau),v_n(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s,$$

$$v_{n+1}(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,v_n(\tau),u_n(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s,$$

$$n = 0,1,2,\cdots,$$

当 $n \to \infty$时,

$$u_n(t) \rightrightarrows u^*(t),\ v_n(t) \rightrightarrows u^*(t).$$

证明　考虑下列积分方程

$$u(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u(\tau),u(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \quad (u \in P_h)$$

正解的存在性,其中 $G(t,s)$ 是(2.3)所示的格林函数. 为此,定义算子 $A: P_h \times P_h \to P$, 其表达式为

$$A(u,v)(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s.$$

对于任意的 $u \in P_h$, $v \in P_h$, 令 $M_{u,v} = \max\Big\{1,\frac{1}{a_u},b_u,\frac{1}{a_v},b_v\Big\}$, 其中 a_u, b_u, a_v, b_v 如(2.9)

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所示,因此 $\frac{1}{M_{u,v}}h(t) \leqslant u(t) \leqslant M_{u,v}h(t)$, $\frac{1}{M_{u,v}}h(t) \leqslant v(t) \leqslant M_{u,v}h(t)$.

首先,证明算子 $A:P_h \times P_h \to P$ 是良定义的. 对于任意的 $u \in P_h$, $v \in P_h$, 由引理 2.2 和条件 (H_1)—(H_3), 有

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leqslant \int_0^1 K(s)t\,\varphi_q\Big(\int_0^s f\Big(\tau,M_{u,v}h(\tau),\frac{1}{M_{u,v}}h(\tau)\Big)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leqslant \int_0^1 K(s)t\,\varphi_q\Big(\int_0^s M_{u,v}{}^{\gamma}f(\tau,h(\tau),h(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &= M_{u,v}{}^{\gamma(q-1)}t\int_0^1 K(s)\,\varphi_q\Big(\int_0^s f(\tau,\tau,\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leqslant M_{u,v}{}^{\gamma(q-1)}t\int_0^1 K(s)\,\varphi_q\Big(\int_0^s f(\tau,1,\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &< +\infty. \end{aligned} \tag{2.10}$$

那么,算子 $A:P_h \times P_h \to P$ 是良定义的.

其次,证明算子 $A:P_h \times P_h \to P_h$. 对于任意的 $u \in P_h$, $v \in P_h$, 取

$$Q_{u,v} > \max\Big\{M_{u,v}{}^{\gamma(q-1)}\Big[\int_0^1 J(s)\,\varphi_q\Big(\int_0^s f(\tau,0,1)\,\mathrm{d}\tau\Big)\mathrm{d}s\Big]^{-1},1,M_{u,v}{}^{\gamma(q-1)}\int_0^1 K(s)\,\varphi_q\Big(\int_0^s f(\tau,1,\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s\Big\}.$$

由(2.10)有

$$\begin{aligned} A(u,v)(t) &\leqslant M_{u,v}{}^{\gamma(q-1)}t\int_0^1 K(s)\,\varphi_q\Big(\int_0^s f(\tau,1,\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\leqslant Q_{u,v}t = Q_{u,v}h(t); \end{aligned}$$

由条件 (H_2) 和引理 2.2,得

$$\begin{aligned} &A(u,v)(t) \\ &= \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geqslant \int_0^1 J(s)t\,\varphi_q\Big(\int_0^s f\Big(\tau,\frac{1}{M_{u,v}}h(\tau),M_{u,v}h(\tau)\Big)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geqslant \int_0^1 J(s)t\,\varphi_q\Big(\int_0^s \Big(\frac{1}{M_{u,v}}\Big)^{\gamma}f(\tau,h(\tau),h(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &= \Big(\frac{1}{M_{u,v}}\Big)^{\gamma(q-1)}t\int_0^1 J(s)\,\varphi_q\Big(\int_0^s f(\tau,\tau,\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geqslant \Big(\frac{1}{M_{u,v}}\Big)^{\gamma(q-1)}t\int_0^1 J(s)\,\varphi_q\Big(\int_0^s f(\tau,0,1)\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geqslant \frac{1}{Q_{u,v}}t = \frac{1}{Q_{u,v}}h(t), \end{aligned}$$

所以,有

$$\frac{1}{Q_{u,v}}h(t) \leqslant A(u,v)(t) \leqslant Q_{u,v}h(t).$$

从而得到算子 $A:P_h \times P_h \to P_h$.

另一方面,验证算子 A 满足引理 1.4 的所有条件.

首先,证明 A 是混合单调算子. 事实上,对于任意的 u_i, $v_i \in P_h$, $i = 1,2$,设 $u_1 \geqslant u_2$, $v_1 \leqslant v_2$. 由条件 (H_1) 和 φ_q 的单调递增性,有

$$\begin{aligned} A(u_1,v_1)(t) &= \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u_1(\tau),v_1(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geqslant \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u_2(\tau),v_2(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &= A(u_2,v_2)(t). \end{aligned}$$

因此, A 是混合单调算子.

对于任意的 $u,v \in P_h$, $r \in (0,1)$, 由条件 (H_2), 可以得到

$$\begin{aligned} &A(ru,r^{-1}v)(t) \\ &= \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,ru(\tau),r^{-1}v(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &\geqslant \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s r^{\gamma}f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \\ &= (r^{\gamma})^{q-1}A(u,v)(t), \end{aligned}$$

设 $\varphi(t) = (t^{\gamma})^{q-1}$, $t \in (0,1)$, 由 $0 < \gamma(q-1) < 1$, 可知对任意的 $t \in (0,1)$, 都有 $\varphi(t) \in (t,1)$. 所以,

$$A(tu,t^{-1}v)(t) \geqslant \varphi(t)A(u,v)(t).$$

从而满足引理 1.4 的所有条件,那么,存在 $u^* \in P_h$, 使得 $A(u^*,u^*) = u^*$. 同时,对于任意的初值 $u_0,v_0 \in P_h$, 序列

$$u_{n+1} = A(u_n,v_n), v_{n+1} = A(v_n,u_n), n = 0,1,2,\cdots,$$

当 $n \to \infty$时,

$$\|u_n - u^*\| \to 0, \|v_n - u^*\| \to 0.$$

更确切地,当 $n \to \infty$时,有

$$u_{n+1}(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,u_n(\tau),v_n(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \rightrightarrows u^*(t),$$

$$v_{n+1}(t) = \int_0^1 G(t,s)\,\varphi_q\Big(\int_0^s f(\tau,v_n(\tau),u_n(\tau))\,\mathrm{d}\tau\Big)\mathrm{d}s \rightrightarrows u^*(t).$$

3 例 子

考虑下列例题

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$$\begin{cases}(\varphi_{\frac{3}{2}}({}^{c}D_{0+}^{\frac{7}{3}}u(t)))' + u^{\frac{1}{4}}\sin t + t^{-\frac{1}{2}}u^{-\frac{1}{5}} + t^{\frac{1}{2}}u^{-\frac{1}{3}} = 0, t \in (0,1),\\ u(0) = u''(0) {}^{c}D_{0+}^{\frac{7}{3}}u(0) = 0,\\ u(1) = \frac{3}{2}\int_0^1 u(s)\,\mathrm{d}s.\end{cases} \tag{3.1}$$

观察(3.1),显然 $p = \frac{3}{2}, \alpha = \frac{7}{3}, \lambda = \frac{3}{2}$ 并且

$$f(t,x,y) = x^{\frac{1}{4}}\sin t + t^{-\frac{1}{2}}y^{-\frac{1}{5}} + t^{\frac{1}{2}}y^{-\frac{1}{3}}.$$

通过简单的计算得 $q = 3$.

可以验证,$f(t,x,y)$ 满足条件 (H_1).

又 $f(t,0,1) = t^{-\frac{1}{2}} + t^{\frac{1}{2}} \neq 0, f(t,1,t) = \sin t + t^{-\frac{7}{10}} + t^{\frac{1}{6}}$, 有

$$\int_0^1 f(t,1,t)\,\mathrm{d}t < +\infty,$$

那么,条件 (H_2) 满足.

下面验证满足条件 (H_3), 取 $\gamma = \frac{1}{3}$, 则 $\gamma \in (0,p-1) = \left(0,\frac{1}{2}\right)$. 因此,对于任意的 $r \in (0,1)$, $x \in [0,+\infty), y \in (0,+\infty)$, 有

$$\begin{aligned} &f(t,rx,r^{-1}y)\\ &= (rx)^{\frac{1}{4}}\sin t + t^{-\frac{1}{2}}(r^{-1}y)^{-\frac{1}{5}} + t^{\frac{1}{2}}(r^{-1}y)^{-\frac{1}{3}}\\ &= r^{\frac{1}{3}}(r^{-\frac{1}{12}}x^{\frac{1}{4}}\sin t + t^{-\frac{1}{2}}r^{-\frac{2}{15}}y^{-\frac{1}{5}} + t^{\frac{1}{2}}y^{-\frac{1}{3}})\\ &\geqslant r^{\frac{1}{3}}(x^{\frac{1}{4}}\sin t + t^{-\frac{1}{2}}y^{-\frac{1}{5}} + t^{\frac{1}{2}}y^{-\frac{1}{3}})\\ &= r^{\frac{1}{3}}f(t,x,y).\end{aligned}$$

因此,它满足定理 2.1 的所有条件. 那么,问题(3.1)有唯一的正解 $u^* \in P_h$, 其中 $h(t) = t, t \in (0,1)$. 并且,对于任意的初值 $u_0, v_0 \in P_h$, 序列

$$u_{n+1}(t) = \int_0^1 G(t,s)\,\varphi_3\left(\int_0^s (u_n^{\frac{1}{4}}\sin\tau + \tau^{-\frac{1}{2}}v_n^{-\frac{1}{5}} + \tau^{\frac{1}{2}}v_n^{-\frac{1}{3}})\,\mathrm{d}\tau\right)\mathrm{d}s,$$

$$v_{n+1}(t) = \int_0^1 G(t,s)\,\varphi_3\left(\int_0^s (v_n^{\frac{1}{4}}\sin\tau + \tau^{-\frac{1}{2}}u_n^{-\frac{1}{5}} + \tau^{\frac{1}{2}}u_n^{-\frac{1}{3}})\,\mathrm{d}\tau\right)\mathrm{d}s,\ n = 0,1,2,\cdots,$$

其中 $G(t,s)$ 是(2.3)定义的格林函数,当 $n \to \infty$ 时, $u_n(t) \rightrightarrows u^*(t), v_n(t) \rightrightarrows u^*(t)$.

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Optical Anti-reflection Properties of Amorphous Vanadium Pentoxide Thin Films Prepared at High Temperature

WANG Qiujin WANG Hai

(Department of Physics, Capital Normal University, Beijing 100048)

Abstract: The amorphous vanadium pentoxide (V_2O_5) thin films were prepared on the Al_2O_3/Si substrates by ultra-high vacuum magnetron sputtering, and the effects of high temperature annealing on the crystal structure, surface morphology and optical properties were studied. The surface of the sample was smooth and dense with continuity, which was showed through scanning electron microscopy. In the visible light range, the film reflectivity was smallest (about 18%) at the wavelength of 497 nm. After high-temperature annealing for 15 minutes, there was no other diffraction peak in the X-ray diffraction spectrum of the sample except the diffraction peak of the substrate, indicating that the sample was still amorphous. The minimum reflectivity of samples after high temperature annealing can be further reduced to about 6%. This work could be helpful for the preparation and application of high temperature optical devices based on V_2O_5 thin films.

Keywords: ultra-high vacuum; high temperature; amorphous V_2O_5 thin films; reflectivity

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Existence and Uniqueness of Solutions for Singular Fractional Boundary Value Problems with p-Laplacian Operator

WANG Li

(School of Mathematical Sciences, Qufu Normal University, Qufu Shandong 273165)

Abstract: In this paper, we study the existence and uniqueness of solutions for singular fractional boundary value problem of Caputo derivative involving the p-Laplacian operator. Our analyses rely on the mixed monotone operator theory and the fixed point results of the related operator equations.

Keywords: singular fractional differential equation; p-Laplacian operator; mixed monotone operator; integral boundary condition

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