DOI:10.19789/j.1004-9398.2022.04.004
文献引用:宋云涛,王慧颖,苏莉.复化三点Gauss-Legendre数值求积公式的外推算法[J].首都师范大学学报(自然科学版),2022,43(4):19-22. SONG Y T, WANG H Y, SU L. Extrapolation method for composite three points Gauss-Legendre numerical integral formulas[J]. Journal of Capital Normal University(Natural Science Edition),2022,43(4):19-22.

复化三点Gauss-Legendre数值求积公式的外推算法

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摘要:通过Richardson外推的方法,对复化三点Gauss-Legendre求积公式外推,得到高精度的数值积分公式——复化三点Gauss-Legendre求积公式序列$\{L_k(h)\}$.
关键词:Richardson外推算法;Gauss-Legendre求积公式;复化求积公式
中图分类号:O172.2

Extrapolation method for composite three points Gauss-Legendre numerical integral formulas

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Abstract: Composite three points Gauss-Legendre integral formula is extrapolated by Richardson extrapolation method, and high precision numerical integral formulas——sequence $\{L_k(h)\}$ of composite three points Gauss-Legendre integral formulas are obtained.
Keywords: Richardson extrapolation method; Gauss-Legendre integral formula; composite integral formula
CLC:O172.2

0 引 言

目前关于积分$\int_a^b f(x)\mathrm{d}x$的复化三点Gauss-Legendre求积公式给出了收敛性,误差为$O(h^6)$[1-2],现有文献没有给出高精度的求积公式,计算往往达不到所需精度要求.为提高计算精度,本文在参考文献[3-9]的基础上,利用Taylor展开方法,推导出复化三点Gauss-Legendre求积公式渐近展开式,再通过Richardson外推,得到高精度的复化三点求积公式系列$\{L_k(h)\}$,收敛阶提高为$O(h^{2k+6})$.

1 问 题[10-12]

对于定积分$\int_a^b f(x)\mathrm{d}x$,把$[a, b]$分成n个相等的小区间,每个小区长度$h=\frac{b-a}{n}$,记节点$x_k=a+kh,(k=0,1,2,\cdots,n)$,$x_{k-\frac{1}{2}}=\frac{1}{2}(x_{k-1}+x_k)$为第$k$个小区间$[x_{k-1}, x_k](k=1, 2,\cdots, n)$的中点,若函数$f(x)\in C^6[a,b]$,则有复化三点Gauss-Legendre求积公式:

收稿日期:2021-05-17
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$$\int_a^b f(x)\mathrm{d}x \approx L(h)=\frac{h}{2}\sum_{k=1}^{n}\left[\frac{5}{9}f\left(x_{k-\frac{1}{2}}-\frac{\sqrt{15}}{10}h\right)+\frac{8}{9}f\left(x_{k-\frac{1}{2}}\right)+\frac{5}{9}f\left(x_{k-\frac{1}{2}}+\frac{\sqrt{15}}{10}h\right)\right]. \tag{1}$$

复化三点Gauss-Legendre求积公式的余项为

$$R(f)=\sum_{k=1}^{n}\frac{h^7}{201\,600}f^{(6)}(\eta_k)=\frac{h^6}{201\,600}\sum_{k=1}^{n}\frac{b-a}{n}f^{(6)}(\eta_k)=\frac{h^6(b-a)}{201\,600}\cdot\frac{1}{n}\sum_{k=1}^{n}f^{(6)}(\eta_k),\ \eta_k\in\left[x_{k-1},x_k\right],\ k=1,2,\cdots,n.$$

因此,需要提高精度.

2 复化三点Gauss-Legendre求积公式渐近展开式[13-16]

设$f(x)\in C^{\infty}[a,b]$,将$f\left(x_{k-\frac{1}{2}}-\frac{\sqrt{15}}{10}h\right)$,$f\left(x_{k-\frac{1}{2}}+\frac{\sqrt{15}}{10}h\right)$在子区间$[x_{k-1},x_k]$中点$x_{k-\frac{1}{2}}$处展开成Taylor级数,得

$$f\left(x_{k-\frac{1}{2}}-\frac{\sqrt{15}}{10}h\right)=f\left(x_{k-\frac{1}{2}}\right)-f'\left(x_{k-\frac{1}{2}}\right)\frac{\sqrt{15}}{10}h+\frac{f''\left(x_{k-\frac{1}{2}}\right)}{2!}\left(\frac{\sqrt{15}}{10}h\right)^2-\frac{f'''\left(x_{k-\frac{1}{2}}\right)}{3!}\left(\frac{\sqrt{15}}{10}h\right)^3+\frac{f^{(4)}\left(x_{k-\frac{1}{2}}\right)}{4!}\left(\frac{\sqrt{15}}{10}h\right)^4-\frac{f^{(5)}\left(x_{k-\frac{1}{2}}\right)}{5!}\left(\frac{\sqrt{15}}{10}h\right)^5+\frac{f^{(6)}\left(x_{k-\frac{1}{2}}\right)}{6!}\left(\frac{\sqrt{15}}{10}h\right)^6-\frac{f^{(7)}\left(x_{k-\frac{1}{2}}\right)}{7!}\left(\frac{\sqrt{15}}{10}h\right)^7+\frac{f^{(8)}\left(x_{k-\frac{1}{2}}\right)}{8!}\left(\frac{\sqrt{15}}{10}h\right)^8-\frac{f^{(9)}\left(x_{k-\frac{1}{2}}\right)}{9!}\left(\frac{\sqrt{15}}{10}h\right)^9+\frac{f^{(10)}\left(x_{k-\frac{1}{2}}\right)}{10!}\left(\frac{\sqrt{15}}{10}h\right)^{10}+\cdots. \tag{2}$$

$$f\left(x_{k-\frac{1}{2}}+\frac{\sqrt{15}}{10}h\right)=f\left(x_{k-\frac{1}{2}}\right)+f'\left(x_{k-\frac{1}{2}}\right)\frac{\sqrt{15}}{10}h+\frac{f''\left(x_{k-\frac{1}{2}}\right)}{2!}\left(\frac{\sqrt{15}}{10}h\right)^2+\frac{f'''\left(x_{k-\frac{1}{2}}\right)}{3!}\left(\frac{\sqrt{15}}{10}h\right)^3+\frac{f^{(4)}\left(x_{k-\frac{1}{2}}\right)}{4!}\left(\frac{\sqrt{15}}{10}h\right)^4+\frac{f^{(5)}\left(x_{k-\frac{1}{2}}\right)}{5!}\left(\frac{\sqrt{15}}{10}h\right)^5+\frac{f^{(6)}\left(x_{k-\frac{1}{2}}\right)}{6!}\left(\frac{\sqrt{15}}{10}h\right)^6+\frac{f^{(7)}\left(x_{k-\frac{1}{2}}\right)}{7!}\left(\frac{\sqrt{15}}{10}h\right)^7+\frac{f^{(8)}\left(x_{k-\frac{1}{2}}\right)}{8!}\left(\frac{\sqrt{15}}{10}h\right)^8+\frac{f^{(9)}\left(x_{k-\frac{1}{2}}\right)}{9!}\left(\frac{\sqrt{15}}{10}h\right)^9+\frac{f^{(10)}\left(x_{k-\frac{1}{2}}\right)}{10!}\left(\frac{\sqrt{15}}{10}h\right)^{10}+\cdots. \tag{3}$$

将式(2)和(3)代入式(1)有

$$L(h)=h\sum_{k=1}^{n}f\left(x_{k-\frac{1}{2}}\right)+\frac{5\times3}{9\times2!\times2\times10}h^3\sum_{k=1}^{n}f''\left(x_{k-\frac{1}{2}}\right)+\frac{5\times3^2}{9\times4!\times2^2\times10^2}h^5\sum_{k=1}^{n}f^{(4)}\left(x_{k-\frac{1}{2}}\right)+\frac{5\times3^3}{9\times6!\times2^3\times10^3}h^7\sum_{k=1}^{n}f^{(6)}\left(x_{k-\frac{1}{2}}\right)+\frac{5\times3^4}{9\times8!\times2^4\times10^4}h^9\sum_{k=1}^{n}f^{(8)}\left(x_{k-\frac{1}{2}}\right)+\frac{5\times3^5}{9\times10!\times2^5\times10^5}h^{11}\sum_{k=1}^{n}f^{(10)}\left(x_{k-\frac{1}{2}}\right)+\frac{5\times3^6}{9\times12!\times2^6\times10^6}h^{13}\sum_{k=1}^{n}f^{(12)}\left(x_{k-\frac{1}{2}}\right)+\cdots. \tag{4}$$

再将函数$f(x)$在子区间$[x_{k-1},x_k]$中点$x_{k-\frac{1}{2}}$处展开成Taylor级数得:

$$f(x)=f\left(x_{k-\frac{1}{2}}\right)+f'\left(x_{k-\frac{1}{2}}\right)\left(x-x_{k-\frac{1}{2}}\right)+\frac{f''\left(x_{k-\frac{1}{2}}\right)}{2!}\left(x-x_{k-\frac{1}{2}}\right)^2+\frac{f'''\left(x_{k-\frac{1}{2}}\right)}{3!}\left(x-x_{k-\frac{1}{2}}\right)^3+\frac{f^{(4)}\left(x_{k-\frac{1}{2}}\right)}{4!}\left(x-x_{k-\frac{1}{2}}\right)^4+\frac{f^{(5)}\left(x_{k-\frac{1}{2}}\right)}{5!}\left(x-x_{k-\frac{1}{2}}\right)^5+\frac{f^{(6)}\left(x_{k-\frac{1}{2}}\right)}{6!}\left(x-x_{k-\frac{1}{2}}\right)^6+\frac{f^{(7)}\left(x_{k-\frac{1}{2}}\right)}{7!}\left(x-x_{k-\frac{1}{2}}\right)^7+\frac{f^{(8)}\left(x_{k-\frac{1}{2}}\right)}{8!}\left(x-x_{k-\frac{1}{2}}\right)^8+\cdots. \tag{5}$$

在$[x_{k-1},x_k]$上分别对式(5)左右两端求积分得

$$\int_{x_{k-1}}^{x_k}f(x)\mathrm{d}x=hf\left(x_{k-\frac{1}{2}}\right)+\frac{h^3}{2!\times2^2\times3}f''\left(x_{k-\frac{1}{2}}\right)+\frac{h^5}{4!\times2^4\times5}f^{(4)}\left(x_{k-\frac{1}{2}}\right)+\frac{h^7}{6!\times2^6\times7}f^{(6)}\left(x_{k-\frac{1}{2}}\right)+\frac{h^9}{8!\times2^8\times9}f^{(8)}\left(x_{k-\frac{1}{2}}\right)+\cdots, \tag{6}$$

于是

$$\int_a^b f(x)\mathrm{d}x = h\sum_{k=1}^{n} f(x_{k-\frac{1}{2}}) + \frac{h^3}{2!\times 2^2\times 3}\sum_{k=1}^{n} f''(x_{k-\frac{1}{2}}) + \frac{h^5}{4!\times 2^4\times 5}\sum_{k=1}^{n} f^{(4)}(x_{k-\frac{1}{2}}) + \frac{h^7}{6!\times 2^6\times 7}\sum_{k=1}^{n} f^{(6)}(x_{k-\frac{1}{2}}) + \frac{h^9}{8!\times 2^8\times 9}\sum_{k=1}^{n} f^{(8)}(x_{k-\frac{1}{2}}) + \frac{h^{11}}{10!\times 2^{10}\times 11}\sum_{k=1}^{n} f^{(10)}(x_{k-\frac{1}{2}}) + \frac{h^{13}}{12!\times 2^{12}\times 13}\sum_{k=1}^{n} f^{(12)}(x_{k-\frac{1}{2}}) + \cdots. \tag{7}$$

由式(4)~(7)得

$$L(h) - \int_a^b f(x)\mathrm{d}x = \left(\frac{5\times 3^3}{9\times 6!\times 2^3\times 10^3} - \frac{1}{6!\times 2^6\times 7}\right)h^7\sum_{k=1}^{n} f^{(6)}(x_{k-\frac{1}{2}}) + \left(\frac{5\times 3^4}{9\times 8!\times 2^4\times 10^4} - \frac{1}{8!\times 2^8\times 9}\right)h^9\sum_{k=1}^{n} f^{(8)}(x_{k-\frac{1}{2}}) + \left(\frac{5\times 3^5}{9\times 10!\times 2^5\times 10^5} - \frac{1}{10!\times 2^{10}\times 11}\right)h^{11}\sum_{k=1}^{n} f^{(10)}(x_{k-\frac{1}{2}}) + \left(\frac{5\times 3^6}{9\times 12!\times 2^6\times 10^6} - \frac{1}{12!\times 2^{12}\times 13}\right)h^{13}\sum_{k=1}^{n} f^{(12)}(x_{k-\frac{1}{2}}) + \cdots. \tag{8}$$

由题设$f(x)\in C^{\infty}[a,b]$,可知$f^{(6)}(x)\in C[a,b]$,令最小值$m=\min\limits_{a\leqslant x\leqslant b} f^{(6)}(x)$,最大值$M=\max\limits_{a\leqslant x\leqslant b} f^{(6)}(x)$,则有$m\leqslant\frac{1}{n}\sum\limits_{k=1}^{n} f^{(6)}(x_{k-\frac{1}{2}})\leqslant M$,根据介值定理,在$[a,b]$上至少有一点$\eta_1$,使得$f^{(6)}(\eta_1)=\frac{1}{n}\sum\limits_{k=1}^{n} f^{(6)}(x_{k-\frac{1}{2}})$.

于是可得到式(8)右边第1项

$$\left(\frac{5\times 3^3}{9\times 6!\times 2^3\times 10^3} - \frac{1}{6!\times 2^6\times 7}\right)h^7\sum_{k=1}^{n} f^{(6)}(x_{k-\frac{1}{2}}) = \left(\frac{5\times 3^3}{9\times 6!\times 2^3\times 10^3} - \frac{1}{6!\times 2^6\times 7}\right)(b-a)h^6 f^{(6)}(\eta_1),$$

同理可得到式(8)中右边第2项

$$\left(\frac{5\times 3^4}{9\times 8!\times 2^4\times 10^4} - \frac{1}{8!\times 2^8\times 9}\right)h^9\sum_{k=1}^{n} f^{(8)}(x_{k-\frac{1}{2}}) = \left(\frac{5\times 3^4}{9\times 8!\times 2^4\times 10^4} - \frac{1}{8!\times 2^8\times 9}\right)(b-a)h^8 f^{(8)}(\eta_2),$$

依此类推,式(8)可表示为

$$L(h) - I = \left(\frac{5\times 3^3}{9\times 6!\times 2^3\times 10^3} - \frac{1}{6!\times 2^6\times 7}\right)(b-a)h^6 f^{(6)}(\eta_1) + \left(\frac{5\times 3^4}{9\times 8!\times 2^4\times 10^4} - \frac{1}{8!\times 2^8\times 9}\right)(b-a)h^8 f^{(8)}(\eta_2) + \left(\frac{5\times 3^5}{9\times 10!\times 2^5\times 10^5} - \frac{1}{10!\times 2^{10}\times 11}\right)(b-a)h^{10} f^{(10)}(\eta_3) + \left(\frac{5\times 3^6}{9\times 12!\times 2^6\times 10^6} - \frac{1}{12!\times 2^{12}\times 13}\right)(b-a)h^{12} f^{(12)}(\eta_4) + \cdots. \tag{9}$$

记$\omega_1 = \left(\frac{5\times 3^3}{9\times 6!\times 2^3\times 10^3} - \frac{1}{6!\times 2^6\times 7}\right)(b-a)f^{(6)}(\eta_1)$,

$\omega_2 = \left(\frac{5\times 3^4}{9\times 8!\times 2^4\times 10^4} - \frac{1}{8!\times 2^8\times 9}\right)(b-a)f^{(8)}(\eta_2)$,

$\omega_3 = \left(\frac{5\times 3^5}{9\times 10!\times 2^5\times 10^5} - \frac{1}{10!\times 2^{10}\times 11}\right)(b-a)f^{(10)}(\eta_3)$,

$\omega_4 = \left(\frac{5\times 3^6}{9\times 12!\times 2^6\times 10^6} - \frac{1}{12!\times 2^{12}\times 13}\right)(b-a)f^{(12)}(\eta_4)$,

…

从而得到复化三点Gauss-Legendre求积公式的渐近展开式为

$$L(h) = I + \omega_1 h^6 + \omega_2 h^8 + \omega_3 h^{10} + \omega_4 h^{12} + \cdots, \tag{10}$$

式中ω_i与h无关.

3 复化三点Gauss-Legendre求积公式的Richardson外推算法[17-19]

记

$$L_0(h) = L(h) = I + \omega_1 h^6 + \omega_2 h^8 + \omega_3 h^{10} + \omega_4 h^{12} + \cdots,$$

则

$$L_0\left(\frac{h}{2}\right) = I + \frac{\omega_1}{2^6}h^6 + \frac{\omega_2}{2^8}h^8 + \frac{\omega_3}{2^{10}}h^{10} + \frac{\omega_4}{2^{12}}h^{12} + \cdots, \tag{11}$$

$2^6\times$式(11)$-$式(10),整理得

$$\frac{2^6\times L_0\left(\frac{h}{2}\right) - L_0(h)}{2^6-1} = I + \frac{1-2^2}{2^2\times(2^6-1)}\omega_2 h^8 + \frac{1-2^4}{2^4\times(2^6-1)}\omega_3 h^{10} + \frac{1-2^6}{2^6\times(2^6-1)}\omega_4 h^{12} + \frac{1-2^8}{2^8\times(2^6-1)}\omega_5 h^{14} + \cdots.$$

记$L_1(h)=\frac{2^6\times L_0\left(\frac{h}{2}\right)-L_0(h)}{2^6-1}$,$\omega_1'=\frac{1-2^2}{2^2\times(2^6-1)}\omega_2$,

$\omega_2' = \frac{1-2^4}{2^4\times(2^6-1)}\omega_3$,$\omega_3' = \frac{1-2^6}{2^6\times(2^6-1)}\omega_4$,

$\omega_4' = \frac{1-2^8}{2^8\times(2^6-1)}\omega_5$,….

则

$$L_1(h) = I + \omega_1' h^8 + \omega_2' h^{10} + \omega_3' h^{12} + \omega_4' h^{14} + \cdots, \tag{12}$$

同理

$$L_1(\frac{h}{2}) = I + \frac{\omega_1'}{2^8} h^8 + \frac{\omega_2'}{2^{10}} h^{10} + \frac{\omega_3'}{2^{12}} h^{12} + \frac{\omega_4'}{2^{14}} h^{14} + \cdots, \tag{13}$$

2^8×式(13)−式(12),整理得

$$\frac{2^8 \times L_1(\frac{h}{2}) - L_1(h)}{2^8 - 1} = I + \frac{1 - 2^2}{2^2 \times (2^8 - 1)} \omega_2' h^{10} + \frac{1 - 2^4}{2^4 \times (2^8 - 1)} \omega_3' h^{12} + \frac{1 - 2^6}{2^6 \times (2^8 - 1)} \omega_4' h^{14} + \frac{1 - 2^8}{2^8 \times (2^8 - 1)} \omega_5' h^{16} + \cdots.$$

记 $L_2(h) = \dfrac{2^8 \times L_1(\frac{h}{2}) - L_1(h)}{2^8 - 1}$, $\omega_1'' = \dfrac{1 - 2^2}{2^2 \times (2^8 - 1)} \omega_2'$, $\omega_2'' = \dfrac{1 - 2^4}{2^4 \times (2^8 - 1)} \omega_3'$, $\omega_3'' = \dfrac{1 - 2^6}{2^6 \times (2^8 - 1)} \omega_4'$, $\omega_4'' = \dfrac{1 - 2^8}{2^8 \times (2^8 - 1)} \omega_5'$, ….

则 $L_2(h) = I + \omega_1'' h^{10} + \omega_2'' h^{12} + \omega_3'' h^{14} + \omega_4'' h^{16} + \cdots$. 依此,经过 k 次 Richardson 外推后,即可得数值积分递推公式

$$L_k(h) = \frac{4^{k+2} \times L_{k-1}(\frac{h}{2}) - L_{k-1}(h)}{4^{k+2} - 1}, \tag{14}$$

又

$$L_k(h) = I + \alpha_1 h^{2k+6} + \alpha_2 h^{2k+8} + \alpha_3 h^{2k+10} + \alpha_4 h^{2k+12} + \cdots, \tag{15}$$

式中 α_i 与 h 无关,所以 $L_k(h) - I = O(h^{2k+6})$.

4 结束语

本文利用 Taylor 展开,对复化三点 Gauss-Legendre 求积公式进行 Richardson 外推,得到复化三点求积公式系列 $\{L_k(h)\}$,收敛阶提高为 $O(h^{2k+6})$,加速了求积公式的收敛性,在提高精度的同时大大简化了积分值计算,便于解决实际问题,具有很强的实用性与广泛的应用性.

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