DOI:10.19789/j.1004-9398.2019.01.002

二项式系数与 Fibonacci 数 $4m$ 次幂的关系*

杨 海** 贾亦真 高晓梅

(西安工程大学理学院,陕西 西安 710048)

摘 要

若$\binom{n}{i}=\frac{n!}{i!\ (n-i)!}$($n,i\in\mathbf{N}^*$ 且 $n\geqslant i$)表示二项式系数,第 l 个 Fibonacci 数为 F_l,其中,l 是非负的整数;对任意正整数 n 和非负整数 k,数列$\left\{\binom{n}{i}\right\}_{i=0}^{n}$和$\{F_{k+i}^p\}_{i=0}^{n}$的卷积为$f(k,p,n)=\binom{n}{0}F_k^p+\binom{n}{1}F_{k+1}^p+\cdots+\binom{n}{n}F_{k+n}^p$.论文利用初等数论方法证明了 $p=4m$($m\in\mathbf{N}^*$)时,等式$f(k,4m,n)=\frac{1}{25^m}[L_{2m}^n\cdot L_{4mk+2mn}+C_{4m}^1(-1)^{k+n+1}L_{2m-1}^nL_{(4m-2)k+(2m-1)n}+C_{4m}^2L_{2m-2}^nL_{(4m-4)k+(2m-2)n}+C_{4m}^3(-1)^{k+n+1}L_{2m-3}^nL_{(4m-6)k+(2m-3)n}+\cdots+C_{4m}^{2m}\cdot 2^n]$成立.

关键词: 二项式系数; Fibonacci 数; $4m$ 次幂; 卷积
中图分类号: O156.7

1 引言及主要结论

若整数 n 和 i 满足关系 $n\geqslant i\geqslant 0$,则

$$\binom{n}{i}=\frac{n!}{i!\ (n-i)!}. \tag{1}$$

其中$\binom{n}{i}$是正整数,表示从 n 个互不相同的元素中选取 i 个元素的组合个数.再结合二项式定理可得

$$(1+x)^n=\sum_{i=0}^{n}\binom{n}{i}x^i. \tag{2}$$

其中,x 为任给的复数,故称$\binom{n}{i}$($n\geqslant i\geqslant 0$)为二项式系数.

设 l 为给定的正整数,则 Fibonacci 数列$\{F_l\}$为

$$F_1=F_2=1;\ F_l=F_{l-1}+F_{l-2}(l\geqslant 3).$$

其中,F_l 表示第 l 个 Fibonacci 数.一直以来,国内外很多数论专家和学者都致力于研究二项式系数和 Fibonacci 数的各种性质,并得到很多关于二项式系数的恒等式[1-5]和包含 Fibonacci 数的恒等式[6-12],大大的丰富了这一领域的研究成果.

若 $k\in\mathbf{N}$,且 $p,n\in\mathbf{N}^*$,则数列$\left\{\binom{n}{i}\right\}_{i=0}^{n}$和$\{F_{k+i}^p\}_{i=0}^{n}$的卷积$f(k,p,n)$为

$$f(k,p,n)=\binom{n}{0}F_k^p+\binom{n}{1}F_{k+1}^p+\cdots+\binom{n}{n}F_{k+n}^p=\sum_{i=0}^{n}\binom{n}{i}F_{k+i}^p. \tag{3}$$

对$f(k,p,n)$问题的研究,晁晶晶[13]、陈小芳[14]、杨海等[15]分别给出了 $p=1$,$p=2$ 和 $p=3$ 时的结论,即证明了

$$f(k,1,n)=F_{k+2n}, \tag{4}$$

$$f(k,2,n)=\begin{cases}5^{n/2}L_{2k+n}, & 2\mid n,\\ 5^{(n-1)/2}F_{2k+n}, & 2\nmid n.\end{cases} \tag{5}$$

$$f(k,3,n)=\begin{cases}\frac{1}{5}(2^nF_{3k+2n}-(-1)^{k+n}3F_{k-n}), & k\geqslant n,\\ \frac{1}{5}(2^nF_{3k+2n}+3F_{n-k}), & k<n.\end{cases} \tag{6}$$

其中 F_{k+2n},F_{2k+n},F_{3k+2n},F_{k-n},F_{n-k}分别指第 $k+2n$,$2k+n$,$3k+2n$,$k-n$ 和 $n-k$ 个 Fibonacci 数,而 L_{2k+n}表示第 $2k+n$ 个 Lucas 数.高晓梅等[16]研究了当 $p=4$ 和 $p=$

收稿日期: 2018-5-22
* 国家自然科学基金(11226038,11371012);陕西省自然科学基金资助项目(2017JM1025);陕西省教育厅专项科研计划项目(17JK0323)
** 通信作者: xpuyhai@163.com

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

8 时的情形,得到两个有趣的恒等式,即

$$f(k,4,n)=\frac{1}{25}(3^nL_{4k+2n}-(-1)^{k+n}4L_{2k+n}+3\cdot 2^{n+1}). \tag{7}$$

$$f(k,8,n)=\frac{1}{625}(7^nL_{8k+4n}-8\cdot 4^n(-1)^{k+n}L_{6k+3n}+28\cdot 3^nL_{4k+2n}-56\cdot(-1)^{k+n}L_{2k+n}+35\cdot 2^{n+1}). \tag{8}$$

其中 L_{4k+2n},L_{2k+n},L_{8k+4n},L_{6k+3n},L_{4k+2n},L_{2k+n} 分别表示第 $4k+2n$,$2k+n$,$8k+4n$,$6k+3n$,$4k+2n$ 和第 $2k+n$ 个 Lucas 数.

本文利用初等数论方法[17-18]研究了当 $p=4m(m\in\mathbf{N}^*)$ 时,$f(k,p,n)$ 的情况,给出了有关二项式系数和 Fibonacci 数 $4m$ 次幂的一个恒等式,即证明了如下定理.

定理 1.1　若 $k\in\mathbf{N}$,且 $p,n\in\mathbf{N}^*$,数列 $\left\{\binom{n}{i}\right\}_{i=0}^{n}$ 与 $\{F_{k+i}^{p}\}_{i=0}^{n}$ 的卷积为 $f(k,p,n)$,则当 $p=4m(m\in\mathbf{N}^*)$ 时

$$\begin{aligned}f(k,4m,n)=\frac{1}{25^m}[&L_{2m}^n\cdot L_{4mk+2mn}+C_{4m}^1(-1)^{k+n+1}L_{2m-1}^nL_{(4m-2)k+(2m-1)n}+\\&C_{4m}^2L_{2m-2}^n\cdot L_{(4m-4)k+(2m-2)n}+C_{4m}^3(-1)^{k+n+1}L_{2m-3}^n\cdot\\&L_{(4m-6)k+(2m-3)n}+\cdots+C_{4m}^{2m}\cdot 2^n].\end{aligned} \tag{9}$$

其中 L_{2m},L_{2m-1},L_{2m-2},$\cdots$ 及 $L_{4mk+2mn}$,$L_{(4m-2)k+(2m-1)n}$,$L_{(4m-4)k+(2m-2)n}$,$\cdots$ 分别表示第 $2m$,$2m-1$,$2m-2$,$\cdots$ 以及第 $4mk+2mn$,$(4m-2)k+(2m-1)n$,$(4m-4)k+(2m-2)n$,$\cdots$ 个 Lucas 数.

任意给定一个正整数 m,如何证明得出二项式系数和 Fibonacci 数 m 次幂的恒等关系式,有待于我们进一步去研究.

2　定理的证明

已知

$$F_l=\frac{1}{\sqrt{5}}(\alpha^l-\beta^l), \tag{10}$$

其中 $l\in\mathbf{N}$,且

$$\alpha=\frac{1}{2}(1+\sqrt{5}),\beta=\frac{1}{2}(1-\sqrt{5}). \tag{11}$$

当 $p=4m(m\in\mathbf{N}^*)$ 时,根据(10)和(11)可得

$$\begin{aligned}F_l^{4m}&=\left(\frac{\alpha^l-\beta^l}{\sqrt{5}}\right)^{4m}\\&=\frac{1}{25^m}[\alpha^{4ml}-C_{4m}^1\alpha^{(4m-1)l}\beta^l+C_{4m}^2\alpha^{(4m-2)l}\beta^{2l}-C_{4m}^3\alpha^{(4m-3)l}\beta^{3l}+\cdots+C_{4m}^{4m-2}\alpha^{2l}\beta^{(4m-2)l}-C_{4m}^{4m-1}\alpha^l\beta^{(4m-1)l}+\beta^{4ml}]\\&=\frac{1}{25^m}[\alpha^{4ml}+\beta^{4ml}-C_{4m}^1(-1)^l\alpha^{(4m-2)l}-C_{4m}^1(-1)^l\beta^{(4m-2)l}+C_{4m}^2\alpha^{(4m-4)l}+C_{4m}^2\beta^{(4m-4)l}-C_{4m}^3(-1)^l\alpha^{(4m-6)l}-C_{4m}^3(-1)^l\beta^{(4m-6)l}+\cdots-C_{4m}^{2m-1}(-1)^l\alpha^{2l}-C_{4m}^{2m-1}(-1)^l\beta^{2l}+C_{4m}^{2m}].\end{aligned} \tag{12}$$

将(3)和(12)结合可知

$$\begin{aligned}f(k,4m,n)=\sum_{i=0}^{n}\binom{n}{i}F_{k+i}^{4m}=\frac{1}{25^m}\Bigg(&\alpha^{4mk}\sum_{i=0}^{n}\binom{n}{i}\alpha^{4mi}+\beta^{4mk}\sum_{i=0}^{n}\binom{n}{i}\beta^{4mi}-\\&C_{4m}^1(-1)^k\alpha^{(4m-2)k}\sum_{i=0}^{n}\binom{n}{i}\cdot(-1)^i\alpha^{(4m-2)i}-\\&C_{4m}^1(-1)^k\beta^{(4m-2)k}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{(4m-2)i}+\\&C_{4m}^2\alpha^{(4m-4)k}\sum_{i=0}^{n}\binom{n}{i}\alpha^{(4m-4)i}+\\&C_{4m}^2\beta^{(4m-4)k}\sum_{i=0}^{n}\binom{n}{i}\beta^{(4m-4)i}-\cdots-\\&C_{4m}^{2m-1}(-1)^k\alpha^{2k}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{2i}-\\&C_{4m}^{2m-1}(-1)^k\beta^{2k}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{2i}+\\&C_{4m}^{2m}\cdot 2^n\Bigg].\end{aligned} \tag{13}$$

又由(2)可以得出

$$\sum_{i=0}^{n}\binom{n}{i}\alpha^{4mi}=(1+\alpha^{4m})^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}\beta^{4mi}=(1+\beta^{4m})^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{(4m-2)i}=(1-\alpha^{4m-2})^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{(4m-2)i}=(1-\beta^{4m-2})^n,$$

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net

$$\sum_{i=0}^{n}\binom{n}{i}\alpha^{(4m-4)i}=(1+\alpha^{4m-4})^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}\beta^{(4m-4)i}=(1+\beta^{4m-4})^n,$$

…

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{6i}=(1-\alpha^6)^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{6i}=(1-\beta^6)^n,\qquad(14)$$

$$\sum_{i=0}^{n}\binom{n}{i}\alpha^{4i}=(1+\alpha^4)^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}\beta^{4i}=(1+\beta^4)^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{2i}=(1-\alpha^2)^n,$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\beta^{2i}=(1-\beta^2)^n.$$

结合(11)和(14)可知

$$1+\alpha^{4m}=L_{2m}\alpha^{2m},1+\beta^{4m}=L_{2m}\beta^{2m},$$

$$1-\alpha^{4m-2}=-L_{2m-1}\alpha^{2m-1},1-\beta^{4m-2}=-L_{2m-1}\beta^{2m-1},$$

$$1+\alpha^{4m-4}=L_{2m-2}\alpha^{2m-2},1+\beta^{4m-4}=L_{2m-2}\beta^{2m-2},$$

…

$$1-\alpha^6=-L_3\alpha^3,1-\beta^6=-L_3\beta^3,$$

$$1+\alpha^4=L_2\alpha^2,1+\beta^4=L_2\beta^2,$$

$$1-\alpha^2=-L_1\alpha=-\alpha,1-\beta^2=-L_1\beta=-\beta.\qquad(15)$$

再把(14)和(15)代入(13),可得出以下恒等式

$$\begin{aligned}f(k,4m,n)=\frac{1}{25^m}\big[&L_{2m}^n\cdot L_{4mk+2mn}+\\&C_{4m}^1(-1)^{k+n+1}L_{2m-1}^nL_{(4m-2)k+(2m-1)n}+\\&C_{4m}^2L_{2m-2}^nL_{(4m-4)k+(2m-2)n}+\\&C_{4m}^3(-1)^{k+n+1}L_{2m-3}^n\cdot\\&L_{(4m-6)k+(2m-3)n}+\cdots+C_{4m}^{2m}\cdot 2^n\big].\end{aligned}$$

定理 1.1 得证.

参 考 文 献

[1] 谭明术. 两个二项式系数及相关恒等式[J]. 西南民族大学学报:自然科学版,2004,30(1):7-11.

[2] GUO V J W, LIN Y J, LIU Y, et al. A q-analogue of Zhang's binomial coefficient identities [J]. Discrete Mathematics, 2009, 309(20): 5913-5919.

[3] WU Y, GAO W. Identities involving powers and inverse of binomial coefficients [J]. Journal of Mathematical Research & Exposition, 2011, 31(6): 1021-1029.

[4] YANG J Z, WANG Y P. A kind of identities for products reciprocals of q-binomial c-oefficients [J]. Chinese Quarterly Journal of Mathematics, 2014, 29(4): 573-582.

[5] MICHAEL D H. An identity involving binomial coefficients [J]. The Mathematical Gazette, 2005, 89: 256-258.

[6] 李桂贞,刘国栋. 一些包含 Fibonacci-Lucas 数的恒等式和同余式[J]. 纯粹数学与应用数学,2006,22(2):238-241.

[7] OHTSUKA H, NAKAMURA S. On the sum of reciprocal Fibonacci numbers [J]. The Fibonacci Quarterly, 2012(2): 153-159.

[8] ZHANG W P. Some identities involving the Fibonacci numbers [J]. The Fibonacci Quarterly, 1997, 35: 225-229.

[9] ZHANG W P. Some identities involving the Fibonacci numbers and Lucas numbers [J]. The Fibonacci Quarterly, 2004, 42: 149-154.

[10] MA R, ZHANG W P. Several identities involving the Fibonacci numbers and Lucas numbers [J]. The Fibonacci Quarterly, 2007, 45: 164-171.

[11] ANDRZEJ W. Some identities for the generalized Fibonacci numbers and the generalized Lucas numbers [J]. Applied Mathematics and Computation, 2013, 219: 5564-5568.

[12] CHOO Y. On the generalizations of Fibonacci identities [J]. Results in Mathematics, 2017, 71(1-2): 347-356.

[13] 晁晶晶. 广义杨辉三角形与 Fibonacci 数列的一个关系[J]. 首都师范大学学报:自然科学版,2011,32(2):11-13.

[14] 陈小芳. Fibonacci 数与杨辉三角形的又一个关系[J]. 首都师范大学学报:自然科学版,2014,35(3):1-2.

[15] 杨海,李博,高晓梅. 二项式系数与 Fibonacci 数立方的一个关系[J]. 纺织高校基础科学学报,2015,28(4):395

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

-398.

[16] 高晓梅，杨海，李博. 二项式系数与 Fibonacci 数四次及八次幂的关系[J]. 西安工程大学学报，2017，31(5)：701-705.

[17] 闵嗣鹤，严士健. 初等数论[M]. 北京：高等教育出版社，2010.

[18] 张文鹏. 初等数论[M]. 西安：陕西师范大学出版社，2007.

A Relation Between Binomial Coefficients and 4m Powers of Fibonacci Numbers

YANG Hai　JIA Yizhen　GAO Xiaomei

(College of Science, Xi´an Polytechnic University, Xi´an Shaanxi　710048)

Abstract

Let $\binom{n}{i}=\frac{n!}{i!\ (n-i)!}(n,i\in\mathbf{N}^*$ and $n\geqslant i)$ be a binomial coefficient. Supposing that l is a nonnegative integer and F_l is the l-th Fibonacci number, furthermore, when n is a fixed positive integer and k is a fixed nonnegative integer, the convolution of the sequence $\left\{\binom{n}{i}\right\}_{i=0}^{n}$ and $\{F_{k+i}^p\}_{i=0}^{n}$ is $f(k,p,n)=\binom{n}{0}F_k^p+\binom{n}{1}F_{k+1}^p+\cdots+\binom{n}{n}F_{k+n}^p$. When $p=4m(m\in\mathbf{N}^*)$, the identity $f(k,4m,n)=\frac{1}{25^m}[L_{2m}^n\cdot L_{4mk+2mn}+C_{4m}^1(-1)^{k+n+1}L_{2m-1}^n L_{(4m-2)k+(2m-1)n}+C_{4m}^2\cdot L_{2m-2}^n L_{(4m-4)k+(2m-2)n}+C_{4m}^3(-1)^{k+n+1}L_{2m-3}^n L_{(4m-6)k+(2m-3)n}+\cdots+C_{4m}^{2m}\cdot 2^n]$ is proved.

Keywords: binomial coefficient; Fibonacci number; 4m-th power; convolution

(C)1994-2020 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.net