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新的广义Ostrowski型不等式的加强

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摘要:为了进一步研究广义Ostrowski型不等式,基于积分恒等式和引入参数求最值的方法,建立了一些绝对连续函数和二阶可微函数的不等式,加强了已有文献给出的涉及函数平均值和区间的广义Ostrowski型不等式.

关键词:Ostrowski型不等式;绝对连续函数;二阶可微函数;双边不等式
中图分类号:O178

Strengthen of new generalized Ostrowski type inequalities

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Abstract: In order to further study the generalized Ostrowski type inequality, based on the integral identities and using the method of introducing parameter to find the minumum, some inequalities involving integral means over end intervals covering the entire interval are established for absolutely continuous functions and twice differentiable functions, strengthening new generalized Ostrowski type inequalities in the existing literatures.

Keywords: Ostrowski type inequality; absolutely continuous function; twice differentiable function; bilateral inequality
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0 引 言

1938年,Ostrowski[1]证明了著名的不等式

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right| \leqslant \frac{(x-a)^2+(b-x)^2}{2(b-a)} M, \quad (1)$$

其中$f:[a,b]\to\mathbf{R}$在$[a,b]$上连续,在(a,b)上可微,且对任意$x\in(a,b)$有$|f'(x)|\leqslant M$.

Ostrowski不等式理论在近似理论、概率论和数值积分中有着广泛应用.自1975年Milovanović[2]给出式(1)的推广以后的几十年来,学者们建立了涉及各种类型积分、各种类型的凸函数、各种空间、高阶导数、权函数和带有扰动的Ostrowski型不等式,有关Ostrowski不等式的改进和推广见文献[2-11].

2002年,通过建立带有Peano型核的Montgomery型积分恒等式(下面的引理1),Cerone[12]建立了一类新的Ostrowski型不等式,与其他文献不同的是,给出了$f(x)$与覆盖区间$[a,b]$的区间$[a,x],[x,b]$上的函数平均值的加权算术平均的偏离的界.

引理1[12] 设$f:[a,b]\to\mathbf{R}$是绝对连续函数,则

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对任意的$\lambda\in[0,1]$和任意$x\in(a,b)$有

$$f(x)-\frac{\lambda}{x-a}\int_a^x f(t)\,\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b f(t)\,\mathrm{d}t=\frac{\lambda}{x-a}\int_a^x(t-a)f'(t)\,\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b(b-t)f'(t)\,\mathrm{d}t.$$

定理1[12]　设$f:[a,b]\to\mathbf{R}$是绝对连续函数,且$\|f'\|_\infty=\operatorname{ess\,sup}_{t\in[a,b]}|f'(t)|<\infty$,则对任意的$\lambda\in[0,1]$和任意$x\in(a,b)$有

$$\left|f(x)-\frac{\lambda}{x-a}\int_a^x f(t)\,\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b f(t)\,\mathrm{d}t\right|\leqslant\frac{\lambda(x-a)+(1-\lambda)(x-a)}{2}\|f'\|_\infty.\tag{2}$$

Qayyum等[13]通过建立带有Peano型核并带有参数的涉及一阶导数的Montgomery型积分恒等式,给出式(2)的推广;Qayyum等[14]通过建立带有加权Peano型核的涉及一阶导数的Montgomery型积分恒等式,给出式(2)的加权推广;Qayyum等[15]通过建立涉及二阶导数的积分恒等式(下面的引理2),在二阶导函数有界的情况下,得到式(2)的带有扰动的推广.

引理2[15]　设$f:[a,b]\to\mathbf{R}$二阶可微,且f''在$[a,b]$上可积,则对任意的$\lambda\in[0,1]$和任意$x\in(a,b)$有

$$\frac{\lambda(x-a)-(1-\lambda)(b-x)}{2}f'(x)-f(x)+\frac{\lambda}{x-a}\int_a^x f(t)\,\mathrm{d}t+\frac{1-\lambda}{b-x}\int_x^b f(t)\,\mathrm{d}t=\frac{\lambda}{2(x-a)}\int_a^x(t-a)^2f''(t)\,\mathrm{d}t+\frac{1-\lambda}{2(b-x)}\int_x^b(b-t)^2f''(t)\,\mathrm{d}t.$$

定理2[15]　设$f:[a,b]\to\mathbf{R}$二阶可微,且$\|f''\|_\infty=\operatorname{ess\,sup}_{t\in[a,b]}|f''(t)|<\infty$,则对任意的$\lambda\in[0,1]$和任意$x\in(a,b)$有

$$\left|\frac{\lambda(x-a)-(1-\lambda)(b-x)}{2}f'(x)-f(x)+\frac{\lambda}{x-a}\int_a^x f(t)\,\mathrm{d}t+\frac{1-\lambda}{b-x}\int_x^b f(t)\,\mathrm{d}t\right|\leqslant\frac{\lambda(x-a)^2+(1-\lambda)(x-a)^2}{6}\|f''\|_\infty.\tag{3}$$

Qayyum等[16]又把式(2)和(3)推广到n阶可微函数;Alshanti[17]通过建立新的带有加权Peano型核的积分恒等式,在二阶导函数有界的情况下又对式(3)作了加权推广;Qayyum等[18]通过建立带有Peano型核并带有参数的涉及二阶导数的Montgomery型积分恒等式,给出式(3)的推广;Qayyum等[19]又把这个结果推广到n阶可微函数;Jiang等[20]还将式(2)推广到时间尺度上.

本文将借助引理1和2中的积分恒等式,仿照Cheng和Sun[21]有关引入参数求最值的方法,在式(2)和(3)的右边都减去了一个非负项,给出式(2)和(3)的加强.为方便起见,引入记号

$$I=f(x)-\frac{\lambda}{x-a}\int_a^x f(t)\,\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b f(t)\,\mathrm{d}t,$$

$$J=f(x)-\frac{\lambda(x-a)-(1-\lambda)(b-x)}{2}f'(x)-\frac{\lambda}{x-a}\int_a^x f(t)\,\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b f(t)\,\mathrm{d}t.$$

1　主要结果

定理3　设$f:[a,b]\to\mathbf{R}$是绝对连续函数,且$\|f'\|_\infty=\operatorname{ess\,sup}_{t\in[a,b]}|f'(t)|<\infty$,则对任意的$\lambda\in[0,1]$和任意的$x\in(a,b)$有

$$-\frac{\sigma}{2}\|f'\|_\infty\left[1-\frac{1}{2}\left(1-\frac{T}{\sigma\|f'\|_\infty}\right)^2\right]\leqslant I\leqslant\frac{\sigma}{2}\|f'\|_\infty\left[1-\frac{1}{2}\left(1+\frac{T}{\sigma\|f'\|_\infty}\right)^2\right],\tag{4}$$

其中$\sigma=\lambda(x-a)+(1-\lambda)(b-x)$,$T=\lambda f(a)+(1-\lambda)f(b)-f(x)$.

证明　对任意常数$\varepsilon\in[0,x-a]$,利用引理1有

$$I+\frac{T}{x-a}\varepsilon=\frac{\lambda}{x-a}\int_a^x(t-a-\varepsilon)f'(t)\,\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b\left(b-t-\frac{b-x}{x-a}\varepsilon\right)f'(t)\,\mathrm{d}t=$$

$$\frac{\lambda}{x-a}\left[\int_a^{a+\varepsilon}(t-a-\varepsilon)f'(t)\,\mathrm{d}t+\int_{a+\varepsilon}^x(t-a-\varepsilon)f'(t)\,\mathrm{d}t\right]-\frac{1-\lambda}{b-x}\left[\int_x^{b-\frac{b-x}{x-a}\varepsilon}\left(b-t-\frac{b-x}{x-a}\varepsilon\right)f'(t)\,\mathrm{d}t+\int_{b-\frac{b-x}{x-a}\varepsilon}^b\left(b-t-\frac{b-x}{x-a}\varepsilon\right)f'(t)\,\mathrm{d}t\right]\leqslant$$

$$\frac{\lambda}{x-a}\left[-\|f'\|_\infty\int_a^{a+\varepsilon}(t-a-\varepsilon)\,\mathrm{d}t+\|f'\|_\infty\int_{a+\varepsilon}^x(t-a-\varepsilon)\,\mathrm{d}t\right]-\frac{1-\lambda}{b-x}\left[-\|f'\|_\infty\int_x^{b-\frac{b-x}{x-a}\varepsilon}\left(b-\frac{b-x}{x-a}\varepsilon-t\right)\mathrm{d}t+\right.$$

$$\|f'\|_\infty\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}\left(b-\frac{b-x}{x-a}\varepsilon-t\right)\mathrm{d}t\Bigg]=$$

$$\frac{\sigma\|f'\|_\infty}{2(x-a)^2}\left[(x-a-\varepsilon)^2+\varepsilon^2\right],$$

$$I\leqslant\frac{\sigma}{(x-a)^2}\|f'\|_\infty(\varepsilon-\varepsilon_1)^2+$$

$$\frac{\sigma}{2}\|f'\|_\infty\left[1-\frac{1}{2}\left(1+\frac{T}{\sigma\|f'\|_\infty}\right)^2\right],\tag{5}$$

其中 $\varepsilon_1=\dfrac{(x-a)\left(T+\sigma\|f'\|_\infty\right)}{2\sigma\|f'\|_\infty}$.

注意到 $T=(1-\lambda)\int_x^b f'(t)\,\mathrm{d}t-\lambda\int_a^x f'(t)\,\mathrm{d}t$,所以有 $|T|\leqslant\sigma\|f'\|_\infty$,从而有 $\varepsilon_1\in[0,x-a]$. 在式(5)中取 $\varepsilon=\varepsilon_1$,则式(4)的右边不等式得证.

因为f是绝对连续函数,且$\|f'\|_\infty=\operatorname{ess\,sup}_{t\in[a,b]}|f'(t)|<\infty$,所以$(-f)$也是绝对连续函数,且$\|(-f)'\|_\infty=\operatorname{ess\,sup}_{t\in[a,b]}|f'(t)|<\infty$,对$(-f)$应用已证结论,则式(4)的左边不等式得证.

推论1 设条件同定理3,则对任意的$\lambda\in[0,1]$和任意的$x\in(a,b)$有

$$|I|\leqslant\frac{\sigma}{2}\|f'\|_\infty\left[1-\frac{1}{2}\left(1-\frac{|T|}{\sigma\|f'\|_\infty}\right)^2\right].\tag{6}$$

特别地,若取$\lambda=\dfrac{1}{2}$,$x=\dfrac{a+b}{2}$,则得

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant$$

$$\frac{b-a}{4}\|f'\|_\infty\left[1-\frac{1}{2}\left(1-\frac{2|T_1|}{(b-a)\|f'\|_\infty}\right)^2\right],$$

其中 $T_1=\dfrac{f(a)+f(b)}{2}-f\left(\dfrac{a+b}{2}\right)$. 式(6)是式(2)的加强.

推论2 设条件同定理3,则对任意的$x\in(a,b)$有

$$-\frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty\left[1-\frac{1}{2}\left(1-\right.\right.$$

$$\left.\left.\frac{(x-a)f(a)+(b-x)f(b)-(b-a)f(x)}{\left((x-a)^2+(b-x)^2\right)\|f'\|_\infty}\right)^2\right]\leqslant$$

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant$$

$$\frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty\left[1-\frac{1}{2}\left(1+\right.\right.$$

$$\left.\left.\frac{(x-a)f(a)+(b-x)f(b)-(b-a)f(x)}{\left((x-a)^2+(b-x)^2\right)\|f'\|_\infty}\right)^2\right].$$

从而有Ostrowski不等式的加强:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant$$

$$\frac{(x-a)^2+(b-x)^2}{2(b-a)}\|f'\|_\infty\left[1-\frac{1}{2}\left(1-\right.\right.$$

$$\left.\left.\frac{|(x-a)f(a)+(b-x)f(b)-(b-a)f(x)|}{\left((x-a)^2+(b-x)^2\right)\|f'\|_\infty}\right)^2\right].$$

证明 在定理3中取$\lambda=\dfrac{x-a}{b-x}$即可得证.

定理4 设$f:[a,b]\to\mathbf{R}$二阶可微,且f''在$[a,b]$上可积,存在常数m和M,$m<M$,使得对于任意$t\in[a,b]$有$m\leqslant f''(t)\leqslant M$,则对任意的$\lambda\in[0,1]$和任意$x\in(a,b)$有

$$-\frac{\tau}{6}M\left[1-\frac{M^2}{(M-m)^2}\left(1-\frac{K}{\tau M}\right)^3\right]\leqslant J\leqslant$$

$$-\frac{\tau}{6}m\left[1-\frac{m^2}{(M-m)^2}\left(1-\frac{K}{\tau m}\right)^3\right],\tag{7}$$

其中

$$\tau=\lambda(x-a)^2+(1-\lambda)(b-x)^2,K=\lambda(x-a)\left(f'(x)-f'(a)\right)+(1-\lambda)(b-x)\left(f'(b)-f'(x)\right).$$

证明 对任意常数$\varepsilon\in[0,x-a]$,利用引理2有

$$J+\frac{K}{2(x-a)}\varepsilon=\frac{\lambda}{2(x-a)}\int_a^x\left[\varepsilon^2-(t-a)^2\right]f''(t)\,\mathrm{d}t+$$

$$\frac{1-\lambda}{2(b-x)}\int_x^b\left[\left(\frac{b-x}{x-a}\right)^2\varepsilon^2-(b-t)^2\right]f''(t)\,\mathrm{d}t=$$

$$\frac{\lambda}{2(x-a)}\left\{\int_a^{a+\varepsilon}\left[\varepsilon^2-(t-a)^2\right]f''(t)\,\mathrm{d}t+\right.$$

$$\int_{a+\varepsilon}^{x}\left[\varepsilon^2-(t-a)^2\right]f''(t)\,\mathrm{d}t\Big\}+$$

$$\frac{1-\lambda}{2(b-x)}\Bigg\{\int_{x}^{b-\frac{b-x}{x-a}\varepsilon}\left[\left(\frac{b-x}{x-a}\right)^2\varepsilon^2-(b-t)^2\right]f''(t)\,\mathrm{d}t+$$

$$\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}\left[\left(\frac{b-x}{x-a}\right)^2\varepsilon^2-(b-t)^2\right]f''(t)\,\mathrm{d}t\Bigg\}\leqslant$$

$$\frac{\lambda}{2(x-a)}\Big\{M\int_{a}^{a+\varepsilon}\left[\varepsilon^2-(t-a)^2\right]\mathrm{d}t+$$

$$m\int_{a+\varepsilon}^{x}\left[\varepsilon^2-(t-a)^2\right]\mathrm{d}t\Big\}+$$

$$\frac{1-\lambda}{2(b-x)}\Bigg\{m\int_{x}^{b-\frac{b-x}{x-a}\varepsilon}\left[\left(\frac{b-x}{x-a}\right)^2\varepsilon^2-(b-t)^2\right]\mathrm{d}t+$$

$$M\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}\left[\left(\frac{b-x}{x-a}\right)^2\varepsilon^2-(b-t)^2\right]f''(t)\,\mathrm{d}t\Bigg\}=$$

$$-\frac{\tau}{6}m+\frac{\tau}{2(x-a)^3}\left[(x-a)m\varepsilon^2+\frac{2}{3}(M-m)\varepsilon^3\right]=:\psi(\varepsilon),$$

$$J\leqslant\psi(\varepsilon)-\frac{K}{2(x-a)^2}\varepsilon^2=:\phi(\varepsilon),\qquad(8)$$

$\phi'(\varepsilon)=\dfrac{\tau}{(x-a)^3}(M-m)\varepsilon(\varepsilon-\varepsilon_2)$, 其中 $\varepsilon_2=\dfrac{x-a}{M-m}\left(\dfrac{K}{\tau}-m\right)$. 因为 $K=\lambda(x-a)\int_a^x f''(t)\,\mathrm{d}t+(1-\lambda)(b-x)\int_x^b f''(t)\,\mathrm{d}t$, 所以有 $\tau m\leqslant K\leqslant\tau M$, 从而有 $\varepsilon_2\in[0,x-a]$. 在式(8)中,取 $\varepsilon=\varepsilon_2$,则式(7)的右边不等式得证.

因为当 $m\leqslant f''(t)\leqslant M$ 时,有 $-M\leqslant(-f(t))''\leqslant-m$,对 $(-f)$ 应用已证结论,则式(7)的左边不等式得证.

推论 3　设条件同定理 4,则对任意的 $\lambda\in[0,1]$ 和任意的 $x\in(a,b)$ 有

$$\left|f(x)-\frac{\lambda(x-a)-(1-\lambda)(b-x)}{2}f'(x)+\frac{\tau}{12}(M+m)-\right.$$

$$\left.\frac{\lambda}{x-a}\int_a^x f(t)\,\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b f(t)\,\mathrm{d}t\right|\leqslant$$

$$\frac{\tau}{12}(M-m)-\frac{\tau}{6(M-m)^2}\left(\frac{M-m}{2}-\left|\frac{M+m}{2}-\frac{K}{\tau}\right|\right)^3.\qquad(9)$$

推论 4　设条件同定理 4,则对任意的 $x\in(a,b)$ 有

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)f'(x)+\right.$$

$$\left.\frac{(x-a)^3+(b-x)^3}{12(b-a)}(M+m)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant$$

$$\frac{(x-a)^3+(b-x)^3}{12(b-a)}(M-m)\left[1-\frac{2}{(M-m)^3}\left(\frac{M-m}{2}-\right.\right.$$

$$\left.\left.\left|\frac{M+m}{2}-\frac{Q}{(x-a)^3+(b-x)^3}\right|\right)^3\right],$$

其中 $Q=(x-a)^2\big(f'(x)-f'(a)\big)+(b-x)^2\big(f'(b)-f'(x)\big)$.

证明　在推论 3 中取 $\lambda=\dfrac{x-a}{b-a}$ 即可得证.

推论 5　设 $f:[a,b]\to\mathbf{R}$ 是二阶可微函数,且 f'' 在 $[a,b]$ 上可积,$\|f''\|_\infty=\operatorname*{ess\,sup}_{t\in[a,b]}|f''(t)|<\infty$,则对任意的 $\lambda\in[0,1]$ 和任意的 $x\in(a,b)$ 有

$$|J|\leqslant\frac{\tau}{6}\|f''\|_\infty\left[1-\frac{1}{4}\left(1-\frac{|K|}{\tau\|f''\|_\infty}\right)^3\right].\qquad(10)$$

特别地,若取 $\lambda=\dfrac{1}{2}$, $x=\dfrac{a+b}{2}$,则得

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leqslant$$

$$\frac{(b-a)^2}{24}\|f''\|_\infty\left[1-\frac{1}{4}\left(1-\frac{|f'(b)-f'(a)|}{(b-a)\|f''\|_\infty}\right)^3\right].$$

式(10)是式(3)的加强.

定理 5　设 $f:[a,b]\to\mathbf{R}$ 可微,且 f' 满足 (m,M)-Lipschitz 条件,即存在常数 m 和 M, $m<M$,使得对于任意 $s,t\in[a,b]$, $s<t$ 有 $m(t-s)\leqslant f'(t)-f'(s)\leqslant M(t-s)$,则对任意的 $\lambda\in[0,1]$ 和任意的 $x\in(a,b)$,有式(7)成立.

证明　对任意常数 $\varepsilon\in[0,x-a]$,有

$$J+\frac{K}{2(x-a)}\varepsilon=\frac{\lambda}{x-a}\Big\{\int_a^{a+\varepsilon}(t-a)\big[f'(t)-$$

$$f'(a)\big]\mathrm{d}t+\int_{a+\varepsilon}^{x}(t-a)\big[f'(t)-f'(x)\big]\mathrm{d}t\Big\}+$$

$$\frac{1-\lambda}{b-x}\Bigg\{\int_x^{b-\frac{b-x}{x-a}\varepsilon}(b-t)\big[f'(x)-f'(t)\big]\mathrm{d}t+$$

$$\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}(b-t)\big[f'(b)-f'(t)\big]\mathrm{d}t\Bigg\}\leqslant$$

$$\frac{\lambda}{x-a}\left\{M\int_a^{a+\varepsilon}(t-a)^2\mathrm{d}t+m\int_{a+\varepsilon}^{x}(t-a)(t-x)\mathrm{d}t\right\}+$$
$$\frac{1-\lambda}{b-x}\left\{m\int_x^{b-\frac{b-x}{x-a}\varepsilon}(b-t)(t-x)\mathrm{d}t+M\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}(b-t)^2\mathrm{d}t\right\},$$

接下来的证明类似于定理4,故略去.

推论6　设条件同定理5,则有式(9)成立.

推论7　设$f:[a,b]\to\mathbf{R}$可微,且f'满足M-Lipschitz条件,即存在常数M,使得对于任意$t,s\in[a,b]$有$|f'(t)-f'(s)|\leqslant M|t-s|$,则有

$$|J|\leqslant\frac{\tau}{6}M\left[1-\frac{1}{4}\left(1-\frac{|K|}{\tau M}\right)^3\right].$$

特别地,若取$\lambda=\frac{1}{2}$,$x=\frac{a+b}{2}$,则得

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leqslant$$
$$\frac{(b-a)^2}{24}M\left[1-\frac{1}{4}\left(1-\frac{|f'(b)-f'(a)|}{(b-a)M}\right)^3\right].$$

定理6　设$f:[a,b]\to\mathbf{R}$二阶可微,且f''在$[a,b]$上可积,存在常数m和M,$m<M$,使得对于任意$t\in[a,b]$有$m\leqslant f''(t)\leqslant M$,则对任意的$\lambda\in[0,1]$和任意$x\in(a,b)$有

$$\max\left\{-\frac{\tau}{6}M,\frac{\tau}{12}m-\frac{L}{2}\right\}\leqslant-\frac{\tau}{6}\left[M-\frac{1}{\sqrt{M-m}}\left(M-\frac{2L}{\tau}\right)^{\frac{3}{2}}\right]\leqslant$$
$$J\leqslant-\frac{\tau}{6}\left[m+\frac{1}{\sqrt{M-m}}\left(\frac{2L}{\tau}-m\right)^{\frac{3}{2}}\right]\leqslant$$
$$\min\left\{-\frac{\tau}{6}m,\frac{\tau}{12}M-\frac{L}{2}\right\},\qquad(11)$$

其中

$$\tau=\lambda(x-a)^2+(1-\lambda)(b-x)^2,L=\big(\lambda(x-a)-(1-\lambda)(b-x)\big)f'(x)+\lambda f(a)+(1-\lambda)f(b)-f(x).$$

证明　对任意常数$\varepsilon\in[0,x-a]$,利用引理2有

$$J+\frac{L}{2(x-a)}\varepsilon=\frac{\lambda}{2(x-a)}\int_a^x(t-a)(a+\varepsilon-t)f''(t)\mathrm{d}t+$$
$$\frac{1-\lambda}{2(b-x)}\int_x^b(b-t)\left(t-b+\frac{b-x}{x-a}\varepsilon\right)f''(t)\mathrm{d}t=$$
$$\frac{\lambda}{2(x-a)}\left[\int_a^{a+\varepsilon}(t-a)(a+\varepsilon-t)f''(t)\mathrm{d}t+\int_{a+\varepsilon}^{x}(t-a)(a+\varepsilon-t)f''(t)\mathrm{d}t\right]+$$
$$\frac{1-\lambda}{2(b-x)}\left[\int_x^{b-\frac{b-x}{x-a}\varepsilon}(b-t)\left(t-b+\frac{b-x}{x-a}\varepsilon\right)f''(t)\mathrm{d}t+\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}(b-t)\left(t-b+\frac{b-x}{x-a}\varepsilon\right)f''(t)\mathrm{d}t\right]\leqslant$$
$$\frac{\lambda}{2(x-a)}\left[M\int_a^{a+\varepsilon}(t-a)(a+\varepsilon-t)\mathrm{d}t+m\int_{a+\varepsilon}^{x}(t-a)(a+\varepsilon-t)\mathrm{d}t\right]+$$
$$\frac{1-\lambda}{2(b-x)}\left[m\int_x^{b-\frac{b-x}{x-a}\varepsilon}(b-t)\left(t-b+\frac{b-x}{x-a}\varepsilon\right)\mathrm{d}t+M\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}(b-t)\left(t-b+\frac{b-x}{x-a}\varepsilon\right)\mathrm{d}t\right],$$
$$J\leqslant-\frac{\tau}{6}m+\frac{\tau}{12(x-a)^3}(M-m)\left[\varepsilon^3-3(x-a)^2\Delta_1\varepsilon\right]=:\theta(\varepsilon).\qquad(12)$$

$\theta'(\varepsilon)=\frac{\tau}{4(x-a)^3}(M-m)(\varepsilon^2-\varepsilon_3^2)$,其中$\varepsilon_3=(x-a)\sqrt{\Delta_1}$. 利用$L=\lambda\int_a^x(t-a)f''(t)\mathrm{d}t+(1-\lambda)\int_x^b(b-t)f''(t)\mathrm{d}t$,可得$\frac{\tau}{2}m\leqslant L\leqslant\frac{\tau}{2}M$,于是有$\Delta_1\in[0,1]$,从而有$\varepsilon_3\in[0,x-a]$.$\theta(\varepsilon)$在$\varepsilon=\varepsilon_3$处取得最小值.在式(12)中,取$\varepsilon=\varepsilon_3$,则式(11)的右边不等式得证.

因为当$m\leqslant f''(t)\leqslant M$时,有$-M\leqslant(-f(t))''\leqslant m$,对$(-f)$应用已证结论,则式(11)的左边不等式得证.

推论8　设条件同定理6,则对任意的$\lambda\in[0,1]$和任意的$x\in(a,b)$有

$$\left|f(x)-\frac{\lambda(x-a)-(1-\lambda)(b-x)}{2}f'(x)+\frac{\tau}{12}(M+m)-\frac{\lambda}{x-a}\int_a^x f(t)\mathrm{d}t-\frac{1-\lambda}{b-x}\int_x^b f(t)\mathrm{d}t\right|\leqslant$$
$$\frac{\tau}{12}(M-m)-\frac{\tau}{6\sqrt{M-m}}\left(\frac{M-m}{2}-\left|\frac{M+m}{2}-\frac{2L}{\tau}\right|\right)^{\frac{3}{2}}.\qquad(13)$$

推论9　设$f:[a,b]\to\mathbf{R}$二阶可微,且f''在$[a,b]$上可积,$\|f''\|_\infty=\operatorname{ess\,sup}_{t\in[a,b]}|f''(t)|<\infty$,则对任意的$\lambda\in[0,1]$和任意的$x\in(a,b)$有

$$|J|\leqslant\frac{\tau\|f''\|_\infty}{6}\left[1-2\left(\frac{1}{2}-\frac{|L|}{\tau\|f''\|_\infty}\right)^{\frac{3}{2}}\right]. \tag{14}$$

式(14)是式(3)的加强.

定理7　设$f:[a,b]\to\mathbf{R}$可微,且f'满足(m,M)-Lipschitz条件,即存在常数m和M,$m<M$,使得对于任意$s,t\in[a,b]$,$s<t$有$m(t-s)\leqslant f'(t)-f'(s)\leqslant M(t-s)$,则对任意的$\lambda\in[0,1]$和任意的$x\in(a,b)$,有式(11)成立.

证明　对任意常数$\varepsilon\in[0,x-a]$,有

$$J+\frac{L}{2(x-a)}\varepsilon=\frac{\lambda}{2(x-a)}\left\{\int_a^{a+\varepsilon}(2a+\varepsilon-2t)\left[f'\left(a+\frac{\varepsilon}{2}\right)-f'(t)\right]\mathrm{d}t+\int_{a+\varepsilon}^{x}(2a+\varepsilon-2t)\left[f'(x)-f'(t)\right]\mathrm{d}t\right\}+$$

$$\frac{1-\lambda}{2(b-x)}\left\{\int_x^{b-\frac{b-x}{x-a}\varepsilon}\left(2b-\frac{b-x}{x-a}\varepsilon-2t\right)\left[f'(x)-f'(t)\right]\mathrm{d}t+\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}\left(2b-\frac{b-x}{x-a}\varepsilon-2t\right)\left[f'\left(b-\frac{b-x}{2(x-a)}\varepsilon\right)-f'(t)\right]\mathrm{d}t\right\}\leqslant$$

$$\frac{\lambda}{2(x-a)}\left[M\int_a^{a+\varepsilon}(2a+\varepsilon-2t)\left(a+\frac{\varepsilon}{2}-t\right)\mathrm{d}t+m\int_{a+\varepsilon}^{x}(2a+\varepsilon-2t)(x-t)\,\mathrm{d}t\right]+$$

$$\frac{1-\lambda}{2(b-x)}\left[m\int_x^{b-\frac{b-x}{x-a}\varepsilon}\left(2b-\frac{b-x}{x-a}\varepsilon-2t\right)(x-t)\,\mathrm{d}t+M\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}\left(2b-\frac{b-x}{x-a}\varepsilon-2t\right)\left(b-\frac{b-x}{2(x-a)}\varepsilon-t\right)\mathrm{d}t\right]=$$

$$\frac{\lambda}{2(x-a)}\left[M\int_a^{a+\varepsilon}(t-a)(a+\varepsilon-t)\,\mathrm{d}t+m\int_{a+\varepsilon}^{x}(t-a)(a+\varepsilon-t)\,\mathrm{d}t\right]+$$

$$\frac{1-\lambda}{2(b-x)}\left[m\int_x^{b-\frac{b-x}{x-a}\varepsilon}(b-t)\left(t-b+\frac{b-x}{x-a}\varepsilon\right)\mathrm{d}t+M\int_{b-\frac{b-x}{x-a}\varepsilon}^{b}(b-t)\left(t-b+\frac{b-x}{x-a}\varepsilon\right)\mathrm{d}t\right],$$

接下来的证明类似于定理6,故略去.

推论10　设条件同定理7,则对任意的$\lambda\in[0,1]$和任意的$x\in(a,b)$,有式(13)成立.

推论11　设$f:[a,b]\to\mathbf{R}$可微,且f'满足M-Lipschitz条件,即存在常数M,使得对于任意$t,s\in[a,b]$有$|f'(t)-f'(s)|\leqslant M|t-s|$,则对任意的$\lambda\in[0,1]$和任意的$x\in(a,b)$有

$$|J|\leqslant\frac{\tau M}{6}\left[1-2\left(\frac{1}{2}-\frac{|L|}{\tau M}\right)^{\frac{3}{2}}\right].$$

2　结束语

本文加强了已有的利用一阶导数对函数值与覆盖整个区间的2个区间上函数平均值组合之差的估计结果,分别在二阶导数有界和一阶导数满足Lipschitz条件的情况下,给出已有扰动的结果的加强.用引入参数求最值的方法,还可以继续做对应于经典Ostrowski不等式的推广研究,将式(2)和式(3)推广到各种类型的积分,以及各种类型凸函数和各种空间等.

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