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星形弹性和粘弹性网络的稳定性*

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摘要：鉴于单一材料的星形弹性网络指数稳定性的缺乏，本文研究了由弹性和粘弹性复合材料构成的星形网络的稳定性，其中弹性弦以星形结构在公共端耦合．利用Lyapunov函数方法，本文给出了该网络指数稳定性的充分条件．最后，数值仿真结果验证了结论的有效性．

关键词：粘弹性；指数稳定性；Lyapunov方法
中图分类号：O231.1

Stability of elastic and viscoelastic star-shaped networks*

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Abstract: In view of the lack of exponential stability of the star-shaped elastic network of a single material, this paper studies the stability of a star-shaped network composed of elastic and viscoelastic composite materials, in which the elastic strings are coupled in a star-shaped structure at the common end. Applying the Lyapunov functional method, this paper provides a sufficient condition for the exponential stability of the network. In the end, numerical simulation results veryfy the correctness of the conclusion.

Keywords: viscoelastic; exponential stability; Lyapunov method
CLC: O231.1

0　引　言

近年来，许多学者对粘弹性系统的稳定性问题进行了深入研究，如文献[1-2]．对于部分粘弹性材料，Rivera和Salvatierra[3]采用Lyapunov方法，证明了解的衰减速度同松弛函数衰减速度一致，类似结果在文献[4]及其参考文献中也得到了证明．此外，网络的稳定性研究也已经取得了较多结果，如文献[5-7]．最近，Han和Xu[8]采用Riesz基方法，研究了Timoshenko梁的星形网络，表明在一定条件下，闭环系统渐近稳定．Guo和Xu[9]采用Riesz基方法，进行了一般树形振动弦网络的渐近稳定性和Riesz基的生成研究．Xu和Yuan等[10]采用系统算子的谱分析方法，进行了Euler-Bernoulli梁的星形网络研究，证明了根向量序列的存在性．

星形网络由弹性和粘弹性2种不同类型的材料组成．由于星形弹性弦网络在能量空间中不是指数稳定的[11]，本文旨在添加粘弹性结构于星形弹性网络的末端，来讨论星形弹性和粘弹性网络是否指数稳定，其中n根弦的星形弹性和粘弹性网络如图1所示．

星形弹性和粘弹性网络方程为：

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$$\begin{cases}\partial_{tt}v_i(x,t)=a\partial_{xx}v_i(x,t), & x\in(0,\lambda\ell_i),t>0,\\ \partial_{tt}u_i(x,t)=b\left(\partial_{xx}u_i(x,t)-\int_{-\infty}^{0}g(\theta)\partial_{xx}u_i(x,t+\theta)2\theta\right), & x\in(\lambda\ell_i,\ell_i),t>0.\end{cases}\tag{1}$$

固定的边界条件和连接条件为:

$$\begin{cases}u_i(\ell_i,t)=0, & t>0,\\ v_i(0,t)=v_j(0,t), & t>0,\\ \sum_{i=1}^{N}\partial_x v_i(0,t)=0, & t>0,\\ v_i(\lambda\ell_i,t)=u_i(\lambda\ell_i,t), & t>0,\\ a\partial_x v_i(\lambda\ell_i,t)=b\left(\partial_x u_i(\lambda\ell_i,t)-\int_{-\infty}^{0}g(\theta)\partial_x u_i(\lambda\ell_i,t+\theta)\mathrm{d}\theta\right), & t>0.\end{cases}\tag{2}$$

初始条件为:

$$\begin{cases}v_i(x,0)=v_{0i}(x),\partial_t v_i(x,0)=v_{1i}(x),x\in(0,\lambda\ell_i),\\ u_i(x,\theta)=h_{0i}(x,\theta),\quad x\in(\lambda\ell_i,\ell_i),\theta\in(-\infty,0).\end{cases}\tag{3}$$

式中:$i,j=1,2,3,\cdots,N$, $\lambda\in(0,1)$是弹性部分和粘弹性部分的分布比例,$v_{0i}(x)$、$v_{1i}(x)$是初始值,$h_{0i}(x,\theta)$是历史函数,a、b是正的材料常数.

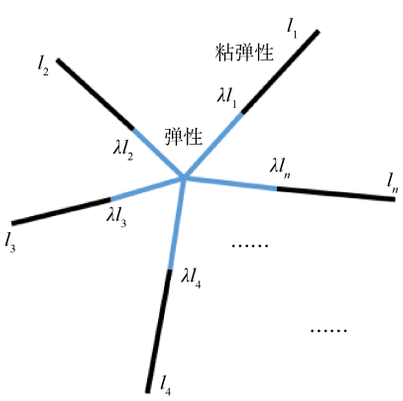


图1 n根弦的星形弹性和粘弹性

1 预备知识

对于松弛函数(g),假设:

(H) $g:R_-\to R_+$是一个单调的连续可微函数,满足条件

$$\|g\|_1=\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta<1,$$

并且存在一个正常数ρ,使得$g'(\theta)>\rho g(\theta)$, $\theta\in R_-$.

如文献[12],设

$$\eta_i(x,\theta,t)=u_i(x,t)-u_i(x,t+\theta),$$
$$q=(1-\|g\|_1),\ i=1,2,\cdots,N.\tag{4}$$

则

$$\begin{cases}\partial_t\eta_i(x,\theta,t)=\partial_\theta\eta_i(x,\theta,t)+z_i(x,t),\ \theta\in R_-,\\ \eta_i(x,0,t)=0,\\ \eta_i(x,\theta,0)=u_i(x,0)-u_i(x,\theta)\end{cases}\tag{5}$$

成立. 其中$\partial_t u_i(x,t)=z_i(x,t)$.

定义函数空间$\Omega=\Omega_1\cup\Omega_2$, $\Omega_1=\bigcup_{i=1}^{N}[0,\lambda\ell_i]$, $\Omega_2=\bigcup_{i=1}^{N}(\lambda\ell_i,\ell_i]H^k[a,b]$是定义在区间$[a,b]$上的$k$阶Sobolev空间,其中$H_B^1[0,\ell_i]=\{f\in H^1[0,\ell_i]\,|\,f(\ell_i)=0\}$, $i=1,2,\cdots,N$.

设函数空间

$$V(\Omega)=\left\{\begin{array}{c}f=(f_1,f_2,\cdots,f_N)^{\mathrm{T}}\in H_B^1[0,\ell_1]\times\\ H_B^1[0,\ell_2]\times\cdots\times H_B^1[0,\ell_N]\\ \left|\,f_i(0)=f_j(0),v_i(\lambda\ell_i,t)=u_i(\lambda\ell_i,t)\right.\end{array}\right\}.$$

对于$f=(f_1,f_2,\cdots,f_N)^{\mathrm{T}}\in V(\Omega)$, 定义$V(\Omega)$中的范数为

$$\|f\|_V=\left(\sum_{i=1}^{N}\int_0^{\ell_i}|f_i'(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

$(V(\Omega),\|\cdot\|_V)$是一个Hilbert空间.

设

$$L^2(\Omega)=L^2[0,\ell_1]\times L^2[0,\ell_2]\times\cdots\times L^2[0,\ell_N].$$

定义范数如下:

$$\|h\|_{L^2}=\left(\sum_{i=1}^{N}\int_0^{\ell_i}|h_i(x)|^2\mathrm{d}x\right)^{\frac{1}{2}},\ h=(h_1,h_2,\cdots,h_N)^{\mathrm{T}}\in L^2(\Omega).$$

显然,$(L^2(\Omega),\|\cdot\|_{L^2})$是一个Hilbert空间.

现定义空间

$$X=V(\Omega)\times L^2(\Omega).$$

具有范数

$$\left\|(f,h)\right\|_X=\left(\left\|f\right\|_V^2+\left\|h\right\|_{L^2}^2\right)^{\frac{1}{2}},(f,h)\in X.$$

可知,X是一个Hilbert空间.

定义符号:

$$v(x)=\left(v_1(x)\ v_2(x)\ \cdots\ v_N(x)\right)^{\mathrm{T}},$$

$$u(x)=\left(u_1(x)\ u_2(x)\cdots\ u_N(x)\right)^{\mathrm{T}},$$

$$\partial_t v(x)=w(x),\ \partial_t u(x)=z(x),$$

$$\eta(x)=\left(\eta_1(x)\ \eta_2(x)\cdots\ \eta_N(x)\right)^{\mathrm{T}}.$$

$$\Delta=\begin{pmatrix}\partial_{xx}&0&\cdots&0\\0&\partial_{xx}&\cdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\cdots&\partial_{xx}\end{pmatrix},\Delta v(x)=\begin{pmatrix}\partial_{xx}v_1(x)\\\partial_{xx}v_2(x)\\\cdots\\\partial_{xx}v_N(x)\end{pmatrix},$$

$$\Delta u(x)=\begin{pmatrix}\partial_{xx}u_1(x)\\\partial_{xx}u_2(x)\\\cdots\\\partial_{xx}u_N(x)\end{pmatrix},\Delta\eta(x)=\begin{pmatrix}\partial_{xx}\eta_1(x)\\\partial_{xx}\eta_2(x)\\\cdots\\\partial_{xx}\eta_N(x)\end{pmatrix}.$$

在X中定义一个算子A如下:

$$AU(x)=\begin{pmatrix}w(x)\\z(x)\\a\Delta v(x)\\b\left(q\Delta u(x)+\int_{-\infty}^{0}g(\theta)\Delta\eta(x)\,\mathrm{d}\theta\right)\end{pmatrix}.\quad(6)$$

其中

$$D(A)=$$

$$\left\{\begin{array}{l}(v,u,w,z)\in X\ \Big|\ v\in H^2(0,\lambda\ell_1)\times\cdots\times H^2(0,\lambda\ell_N),\\\sum\limits_{i=1}^{N}\partial_x v_i(0,t)=0,u\in H^2(\lambda\ell_1,\ell_1)\times\cdots\times H^2(\lambda\ell_N,\ell_N),\\a\partial_x v_i(\lambda\ell_i,t)=b\left(q\partial_x u_i(\lambda\ell_i,t)+\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(\lambda\ell_i,t)\,\mathrm{d}\theta\right),\\w\in V(\Omega_1),\ z\in V(\Omega_2),\ i=1,2,\cdots,N.\end{array}\right\}$$

设 $U(t)=\left(v(x,t)\ u(x,t)\ w(x,t)\ z(x,t)\right)^{\mathrm{T}}$, 则 $U(t)$满足

$$\begin{cases}U_t=AU(t),\ t>0,\\U(0)=U_0.\end{cases}\quad(7)$$

其中$U_0\in X$是初始值.参考文献[13-16],可得波网络(1)~(3)是适定的.

2　稳定性

本节主要研究波网络(1)~(3)的稳定性.由于积分项的存在,本文利用Lyapunov函数方法.令

$$v(x,t)=(v_1(x,t),v_2(x,t),\cdots,v_N(x,t))^{\mathrm{T}},$$

$$u(x,t)=(u_1(x,t),u_2(x,t),\cdots,u_N(x,t))^{\mathrm{T}}$$

是波网络(1)~(3)的一个解,并且

$$\partial_t v(x,t)=w(x,t)=$$

$$(\partial_t v_1(x,t),\partial_t v_2(x,t),\cdots,\partial_t v_N(x,t))^{\mathrm{T}}=$$

$$\left(w_1(x,t),w_2(x,t),\cdots,w_N(x,t)\right)^{\mathrm{T}},\partial_t u(x,t)=$$

$$z(x,t)=(\partial_t u_1(x,t),\partial_t u_2(x,t),\cdots,\partial_t u_N(x,t))^{\mathrm{T}}=$$

$$\left(z_1(x,t),z_2(x,t),\cdots,z_N(x,t)\right)^{\mathrm{T}}.$$

2.1　解的有界性

定义波网络(1)~(3)的能量函数

$$\begin{aligned}E(t)=&\frac{a}{2}\sum_{i=1}^{N}\int_0^{\lambda\ell_i}\left|\partial_x v_i(x,t)\right|^2\mathrm{d}x+\\&\frac{b}{2}\sum_{i=1}^{N}q\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x u_i(x,t)\right|^2\mathrm{d}x+\\&\frac{1}{2}\sum_{i=1}^{N}\int_0^{\lambda\ell_i}\left|w_i(x,t)\right|^2\mathrm{d}x+\frac{1}{2}\sum_{i=1}^{N}q\int_{\lambda\ell_i}^{\ell_i}\left|z_i(x,t)\right|^2\mathrm{d}x+\\&\frac{b}{2}\sum_{i=1}^{N}\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x.\end{aligned}\quad(8)$$

显然,$\dfrac{(1-\left\|g\right\|_1)}{2}\left\|U(t)\right\|_X^2\leqslant E(t)$成立.

命题2.1　设$E(t)$被定义如式(8).则

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=-\frac{b}{2}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\int_{-\infty}^{0}g'(\theta)\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}\theta\mathrm{d}x<0\quad(9)$$

成立.

证明　设$E(t)$被定义如式(8).计算

$$\begin{aligned}\frac{\mathrm{d}E(t)}{\mathrm{d}t}=&a\sum_{i=1}^{N}\left[w_i(x,t)\partial_x v_i(x,t)\right]_0^{\lambda\ell_i}+\\&b\sum_{i=1}^{N}q\left[z_i(x,t)\partial_x u_i(x,t)\right]_{\lambda\ell_i}^{\ell_i}+\\&b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}z_i(x,t)\int_{-\infty}^{0}g(\theta)\partial_{xx}\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}t+\\&b\sum_{i=1}^{N}\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\partial_x\eta_i(x,\theta,t)\partial_x[z_i(x,t)+\\&\partial_\theta\eta_i(x,\theta,t)]\mathrm{d}x\\&(\text{使用边界条件和连接条件})=\\&-\frac{b}{2}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\int_{-\infty}^{0}g'(\theta)\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}\theta\mathrm{d}x.\end{aligned}$$

则式(9)成立.

使用命题2.1,可以得到波网络(1)~(3)解的有界性.

定理2.1　设g满足条件(H),$U(t)=\left(v(x,t),u(x,t),w(x,t),z(x,t)\right)^{\mathrm{T}}$是(1)~(3)的解.

则

$$\|U(t)\|_X^2 \leqslant \frac{2}{1-\|g\|_1}E(t) \leqslant \frac{2}{1-\|g\|_1}E(0),$$
$$\forall t \geqslant 0$$

成立.

2.2 解的衰减

在上一小节中,得到了波网络(1)~(3)解的有界性.在本小节中,将讨论解的衰减性质.为此,构造新的Lyapunov泛函.

首先,定义一个函数

$$E_1(t)=\sum_{i=1}^{N}\int_0^{\lambda\ell_i} v_i(x,t)w_i(x,t)\mathrm{d}x+\sum_{i=1}^{N}\int_{\lambda\ell}^{\ell_i} u_i(x,t)z_i(x,t)\mathrm{d}x-2\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i} z_i(x,t)\int_{-\infty}^{0} g(\theta)(u_i(x,t)-u_i(x,t+\theta))\mathrm{d}\theta\mathrm{d}x. \tag{10}$$

式中:$w_i(x,t)=\partial_t v_i(x,t)$, $z_i(x,t)=\partial_t u_i(x,t)$.

命题 2.2 设$E_1(t)$定义如前所述.则对充分小的$\delta>0$,

$$\begin{aligned}\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} \leqslant{}& \sum_{i=1}^{N}\int_0^{\lambda\ell_i}|w_i(x,t)|^2\mathrm{d}x+(1+\delta-2\|g\|_1)\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}|z_i(x,t)|^2\mathrm{d}x-\\ &a\sum_{i=1}^{N}\int_0^{\lambda\ell_i}|\partial_x v_i(x,t)|^2\mathrm{d}x-\left((1-\delta)bq-\frac{\|g\|_1\delta b}{2}\right)\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}|\partial_x u_i(x,t)|^2\mathrm{d}x+\\ &\left(\frac{b}{2\delta}+\frac{\|g\|_1}{\delta}+2b\|g\|_1\right)\sum_{i=1}^{N}\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}|\partial_x\eta_i(x,\theta,t)|^2\mathrm{d}x+\\ &\frac{g(0)C_p^2}{\delta}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\int_{-\infty}^{0}g'(\theta)|\partial_x\eta_i(x,\theta,t)|^2\mathrm{d}\theta\mathrm{d}x.\end{aligned} \tag{11}$$

证明 设$E_1(t)$被定义如式(10).计算

$$\begin{aligned}\frac{\mathrm{d}E_1(t)}{\mathrm{d}t}={}&\sum_{i=1}^{N}\int_0^{\lambda\ell_i}|w_i(x,t)|^2\mathrm{d}x+\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}|z_i(x,t)|^2\mathrm{d}x+\\ &a[v_i(x,t)\partial_x v_i(x,t)]_0^{\lambda\ell_i}-a\sum_{i=1}^{N}\int_0^{\lambda\ell_i}|\partial_x v_i(x,t)|^2\mathrm{d}x+\\ &b\sum_{i=1}^{N}[qu_i(x,t)\partial_x u_i(x,t)]_{\lambda\ell_i}^{\ell_i}+b\sum_{i=1}^{N}[u_i(x,t)\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta]_{\lambda\ell_i}^{\ell_i}-\\ &b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}q|\partial_x u_i(x,t)|^2\mathrm{d}x-b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\partial_x u_i(x,t)\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x-\\ &2bq\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\partial_{xx}u(x,t)\int_{-\infty}^{0}g(\theta)\eta(x,\theta,t)\mathrm{d}\theta\mathrm{d}x-\\ &2b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left(\int_{-\infty}^{0}g(\theta)\partial_{xx}\eta(x,\theta,t)\mathrm{d}\theta\right)\left(\int_{-\infty}^{0}g(\theta)\eta(x,\theta,t)\mathrm{d}\theta\right)\mathrm{d}x-\\ &2\|g\|_1\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}|z_i(x,t)|^2\mathrm{d}x+2\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}z_i(x,t)\int_{-\infty}^{0}g'(\theta)\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x=\\ &-av_1(0,t)\sum_{i=1}^{N}\partial_x v_i(0,t)+\sum_{i=1}^{N}\int_0^{\lambda\ell_i}|w_i(x,t)|^2\mathrm{d}x+\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}|z_i(x,t)|^2\mathrm{d}x-\\ &a\sum_{i=1}^{N}\int_0^{\lambda\ell_i}|\partial_x v_i(x,t)|^2\mathrm{d}x-b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}q|\partial_x u_i(x,t)|^2\mathrm{d}x-\\ &b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\partial_x u_i(x,t)\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x+\\ &2bq\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\partial_x u_i(x,t)\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x+\\ &2b\|g\|_1\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\int_{-\infty}^{0}g(\theta)|\partial_x\eta_i(x,\theta,t)|^2\mathrm{d}\theta\mathrm{d}x-\\ &2\|g\|_1\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}|z_i(x,t)|^2\mathrm{d}x+2\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}z_i(x,t)\int_{-\infty}^{0}g'(\theta)\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x.\end{aligned}$$

设$\delta>0$充分小.通过Young不等式,得到

$$-b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\partial_x u_i(x,t)\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x\leqslant$$

$$\frac{\|g\|_1\delta}{2}b\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x u_i(x,t)\right|^2\mathrm{d}x+\frac{b}{2\delta}\sum_{i=1}^{N}\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x$$

和

$$2bq\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\partial_x u_i(x,t)\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x\leqslant$$

$$\delta bq\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x u_i(x,t)\right|^2\mathrm{d}x+\frac{\|g\|_1}{\delta}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\int_{-\infty}^{0}g(\theta)\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}\theta\mathrm{d}x,$$

对最后一项,估计

$$2\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}z_i(x,t)\int_{-\infty}^{0}g'(\theta)\eta_i(x,\theta,t)\mathrm{d}\theta\mathrm{d}x\leqslant$$

$$\delta\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left|z_i(x,t)\right|^2\mathrm{d}x+\frac{g(0)C_p^2}{\delta}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\int_{-\infty}^{0}g'(\theta)\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}\theta\mathrm{d}x,$$

其中C_p^2是庞加莱系数.因此,不等式(11)成立.

引进函数

$$\begin{cases}\xi_i(x)=k\dfrac{1-\lambda}{2\lambda}x, & x\in\left(0,\lambda\ell_i\right),k>0,\\ \eta_i(x)=b\left(\dfrac{1+\lambda}{2}\ell_i-x\right), & x\in\left(\lambda\ell_i,\ell_i\right),\end{cases}\tag{12}$$

其中$k\leqslant\min\{a,b\}$.

定义函数

$$E_2(t)=\sum_{i=1}^{N}\int_0^{\lambda\ell_i}\xi_i(x)w_i(x,t)\partial_x v_i(x,t)\mathrm{d}x+\sum_{i=1}^{N}q\int_{\lambda\ell_i}^{\ell_i}\eta_i(x)z_i(x,t)\partial_x u_i(x,t)\mathrm{d}x+$$
$$\sum_{i=1}^{N}\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\eta_i(x)z_i(x,t)\partial_x\eta_i(x,\theta,t)\mathrm{d}x.\tag{13}$$

命题2.3　设$E_2(t)$被定义如式(13).则

$$\frac{\mathrm{d}E_2(t)}{\mathrm{d}t}\leqslant-k\frac{1-\lambda}{4\lambda}\sum_{i=1}^{N}\int_0^{\lambda\ell_i}w_i^2(x,t)\mathrm{d}x+\left(\frac{b}{2}+\frac{b^2l^2g(0)}{8}\right)\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}z_i^2(x,t)\mathrm{d}x-$$
$$ka\frac{1-\lambda}{4\lambda}\sum_{i=1}^{N}\int_0^{\lambda\ell_i}\left|\partial_x v_i(x,t)\right|^2\mathrm{d}x+b^2q^2\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x u_i(x,t)\right|^2\mathrm{d}x+\tag{14}$$
$$b^2\|g\|_1\sum_{i=1}^{N}\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x+\frac{1}{2}\sum_{i=1}^{N}\int_{-\infty}^{0}g'(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x.$$

证明　计算

$$\frac{\mathrm{d}E_2(t)}{\mathrm{d}t}=\frac{1}{2}\sum_{i=1}^{N}\left[\xi_i(x)w_i^2(x,t)\right]_0^{\lambda\ell_i}+\frac{1}{2}\sum_{i=1}^{N}\left[\eta_i(x)z_i^2(x,t)\right]_{\lambda\ell_i}^{\ell_i}+\frac{a}{2}\sum_{i=1}^{N}\xi\left[{}_i(x)\left|\partial_x v_i(x,t)\right|^2\right]_0^{\lambda\ell_i}+$$
$$\frac{b}{2}\sum_{i=1}^{N}\left[\eta_i(x)\left(q\partial_x u_i(x,t)+\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\right)^2\right]_{\lambda\ell_i}^{\ell_i}-$$
$$\frac{1}{2}\sum_{i=1}^{N}\int_0^{\lambda\ell_i}\xi_i'(x)w_i^2(x,t)\mathrm{d}x-\frac{1}{2}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\eta_i'(x)z_i^2(x,t)\mathrm{d}x-\frac{a}{2}\int_0^{\lambda\ell_i}\xi_i'(x)\left|\partial_x v_i(x,t)\right|^2\mathrm{d}x-$$
$$\frac{b}{2}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\eta_i'(x)\left(q\partial_x u_i(x,t)+\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\right)^2\mathrm{d}x-$$

$$\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\eta_i(x)z_i(x,t)\left(\int_{-\infty}^{0}g'(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\right)\mathrm{d}x\leqslant$$

$$-\frac{k}{2}\sum_{i=1}^{N}\int_{0}^{\lambda\ell_i}\frac{1-\lambda}{2\lambda}w_i^2(x,t)\mathrm{d}x+\frac{b}{2}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}z_i^2(x,t)\mathrm{d}x-\frac{ka}{2}\int_{0}^{\lambda\ell_i}\frac{1-\lambda}{2\lambda}\left|\partial_x v_i(x,t)\right|^2\mathrm{d}x+$$

$$\frac{b^2}{2}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left(q\partial_x u_i(x,t)+\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\right)^2\mathrm{d}x-$$

$$\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\eta_i(x)z_i(x,t)\left(\int_{-\infty}^{0}g'(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\right)\mathrm{d}x.$$

估计

$$\int_{\lambda\ell_i}^{\ell_i}\left(q\partial_x u_i(x,t)+\int_{-\infty}^{0}g(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\right)^2\mathrm{d}x\leqslant$$

$$2q^2\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x u_i(x,t)\right|^2\mathrm{d}x+2\left\|g\right\|_1\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x$$

和

$$-\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\eta_i(x)z_i(x,t)\left(\int_{-\infty}^{0}g'(\theta)\partial_x\eta_i(x,\theta,t)\mathrm{d}\theta\right)\mathrm{d}x\leqslant$$

$$\frac{b^2l^2g(0)}{8}\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left|z_i(x,t)\right|^2\mathrm{d}x+\frac{1}{2}\sum_{i=1}^{N}\int_{-\infty}^{0}g'(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x,$$

式中:$\ell=\max\left\{\ell_i,i=1,2,\cdots,N\right\}$. 因此,得到不等式(14)成立,证明完成.

最后,构造函数

$$L(t)=\gamma E(t)+\gamma_1E_1(t)+\gamma_2E_2(t),\tag{15}$$

式中:γ、γ_1、γ_2是正常数.

应用命题2.1~2.3的结果,有估计

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t}=\gamma\frac{\mathrm{d}E(t)}{\mathrm{d}t}+\gamma_1\frac{\mathrm{d}E_1(t)}{\mathrm{d}t}+\gamma_2\frac{\mathrm{d}E_2(t)}{\mathrm{d}t}\leqslant$$

$$\left(\gamma_1-\gamma_2k\frac{1-\lambda}{4\lambda}\right)\sum_{i=1}^{N}\int_{0}^{\lambda\ell_i}w_i^2(x,t)\mathrm{d}x-\left(\alpha\gamma_1+\gamma_2\frac{ka}{2}\frac{1-\lambda}{2\lambda}\right)\sum_{i=1}^{N}\int_{0}^{\lambda\ell_i}\left|\partial_x v_i(x,t)\right|^2\mathrm{d}x+$$

$$\left(-\gamma_1(2\left\|g\right\|_1-1-\delta)+\gamma_2\left(\frac{b}{2}+\frac{b^2\ell^2g(0)}{8}\right)\right)\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}z_i^2(x,t)\mathrm{d}x+$$

$$\left(-\gamma_1((1-\delta)bq-\frac{\left\|g\right\|_1\delta b}{2})+\gamma_2b^2q^2\right)\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x u_i(x,t)\right|^2\mathrm{d}x+$$

$$\left(\gamma_1\left(\frac{b}{2\delta}+\frac{\left\|g\right\|_1}{\delta}+2b\left\|g\right\|_1\right)+\gamma_2b^2\left\|g\right\|_1\right)\int_{-\infty}^{0}g(\theta)\mathrm{d}\theta\sum_{i=1}^{N}\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x+$$

$$\left(\gamma_1\frac{g(0)C_p^2}{\delta}+\frac{1}{2}\gamma_2-\frac{b}{2}\gamma\right)\sum_{i=1}^{N}\int_{-\infty}^{0}g'(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x.$$

使用假设(H),则对于$\frac{b}{2}\gamma-\gamma_1\frac{g(0)C_p^2}{\delta}-\frac{1}{2}\gamma_2>0$,使得

$$\left(\gamma_1\frac{g(0)C_p^2}{\delta}+\frac{1}{2}\gamma_2-\frac{b}{2}\gamma\right)\sum_{i=1}^{N}\int_{-\infty}^{0}g'(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x\leqslant$$

$$-\rho\left(\frac{b}{2}\gamma-\gamma_1\frac{g(0)C_p^2}{\delta}-\frac{1}{2}\gamma_2\right)\sum_{i=1}^{N}\int_{-\infty}^{0}g'(\theta)\mathrm{d}\theta\int_{\lambda\ell_i}^{\ell_i}\left|\partial_x\eta_i(x,\theta,t)\right|^2\mathrm{d}x$$

成立.

定理 2.2　设 $k=\min\{a,b\}$, $\ell=\max\{\ell_i, i=1,2,\cdots,N\}$, 且 $\delta>0$ 充分小. 此外, 设 $g(\theta)$ 满足条件(H). 如果星形网络结构满足

$$0<\lambda<\lambda_0. \tag{16}$$

式中

$$\lambda_0=\min\left\{\frac{kb(2(1-\delta)+\delta\|g\|_1-2\|g\|_1)}{kb(2(1-\delta)+\delta\|g\|_1-2\|g\|_1)+8b^2(1-\|g\|_1)^2}, \frac{k(2\|g\|_1-1-\delta)}{k(2\|g\|_1-1-\delta)+2(b+\frac{b^2\ell^2g(0)}{4})}\right\}<1$$

则系统(1)~(3)的能量函数指数衰减.

证明　为了使 $L(t)$ 前面的系数都小于0, 得到

$$\begin{cases}\gamma_1<\gamma_2 k\dfrac{1-\lambda}{4\lambda},\\ -\gamma_1(2\|g\|_1-1-\delta)+\gamma_2(\dfrac{b}{2}+\dfrac{b^2\ell^2g(0)}{8})<0,\\ -\gamma_1((1-\delta)bq-\dfrac{\|g\|_1\delta b}{2})+\gamma_2b^2q^2<0,\\ \gamma_1(\dfrac{b}{2\delta}+\dfrac{\|g\|_1}{\delta}+2b\|g\|_1)+\gamma_2b^2\|g\|_1-\rho(\dfrac{b}{2}\gamma-\gamma_1\dfrac{g(0)C_p^2}{\delta}-\dfrac{1}{2}\gamma_2)<0.\end{cases} \tag{17}$$

则有

$$\begin{cases}k\dfrac{1-\lambda}{4\lambda}\cdot\dfrac{2bq(1-\delta)-\|g\|_1\delta b}{2b^2q^2}>1,\\ \dfrac{b+\dfrac{b^2\ell^2g(0)}{4}}{2(2\|g\|_1-1-\delta)}<k\dfrac{1-\lambda}{4\lambda}.\end{cases} \tag{18}$$

设 $\delta>0$ 充分小, 并且选择 $\frac{1}{2}<\|g\|_1<1$ 使得 $2\left(2\|g\|_1-1-\delta\right)>0$, 通过解不等式(18), 得到式(16). 选定 γ_1、γ_2, 解式(17)的最后一个不等式, 得到

$$\gamma>\gamma_1\left(\frac{2g(0)C_p^2}{b\delta}+\frac{2}{\rho b}\left(\frac{b}{2\delta}+\frac{\|g\|_1}{\delta}+2b\|g\|_1\right)\right)+\gamma_2\left(\frac{1}{b}+\frac{2b\|g\|_1}{\rho}\right).$$

由以上计算过程, 可知存在正常数 α, 使得

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t}\leqslant-\alpha E(t). \tag{19}$$

另一方面, 通过函数 $E_1(t)$、$E_2(t)$ 和 $E(t)$ 的定义, 当 γ 足够大时, 存在一个正常数 β, 满足 $\left|\gamma_1E_1(t)+\gamma_2E_2(t)\right|<\beta E(t)$. 这表明

$$mE(t)\leqslant L(t)\leqslant ME(t). \tag{20}$$

式中: $m=\gamma-\beta$ 和 $M=\gamma+\beta$. 由式(19)和(20), 可得 $L(t)\leqslant L(0)\mathrm{e}^{-\frac{\alpha}{M}t}, \forall t\geqslant0$. 因此, 有 $E(t)\leqslant\frac{L(0)}{m}\mathrm{e}^{-\frac{\alpha}{M}t}, \forall t\geqslant0$. 证毕.

作为定理2.2的一个应用, 给出波网络解的渐近行为.

推论 2.1　设 $U(t)=(v(x,t), u(x,t), \partial_tv(x,t), \partial_tu(x,t))$ 是波网络(1)~(3)的解. 假设 g 满足条件(H), 并且星形网络的结构满足式(16). 则存在2个正常数 K 和 κ, 使得

$$\|U(t)\|_X^2\leqslant K\mathrm{e}^{-\kappa t}, \forall t\geqslant0.$$

3　数值仿真

在MatLab环境下, 时间上采用隐式欧拉法(时间步长 dt=0.02), 空间上采用二阶中心差分法(空间网格尺寸 N=40). 为简单起见, 假设星形网络由3条边组成, 其每条边波的长度为

$$\ell_1=1, \ell_2=1.5, \ell_3=2.$$

则 $\ell=\max\{1, 1.5, 2\}=2$.

选择 $a=5, b=2, \lambda=0.6<1, v_0(x)=\sin\left(\frac{\pi}{\ell_i}x\right)$ 和 $g(\theta)=\frac{1}{2}\mathrm{e}^{\theta}, \theta\in R_-$, $\|g\|_1<1$ 满足条件(H), 不同长度的波稳定性数值仿真结果如图2所示. 图2表示长度分别为 $\ell_1=1, \ell_2=1.5, \ell_3=2$ 的波随着时间 t 的增长逐渐趋于稳定.

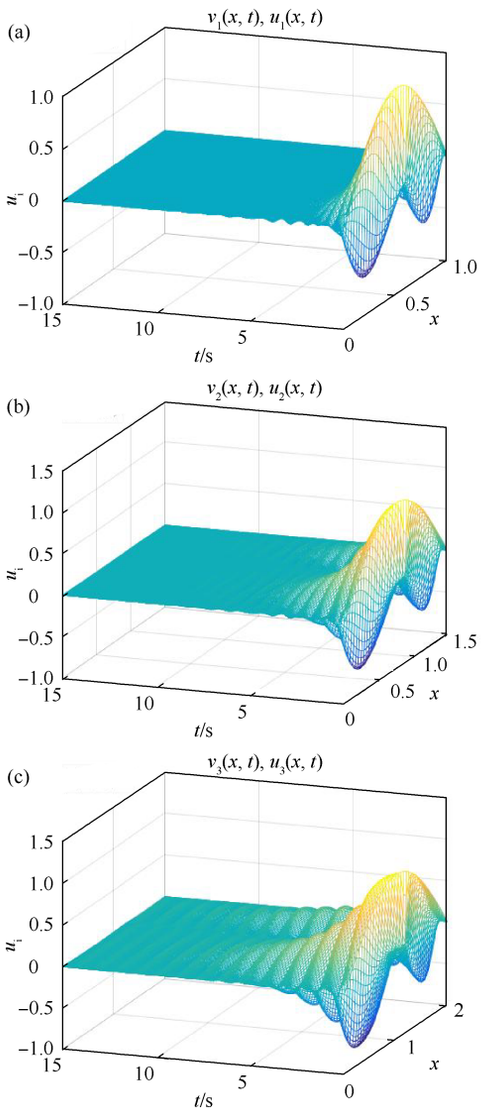


图2　不同波长的波的稳定性

(a)$\ell_1 = 1$;(b)$\ell_2 = 1.5$;(c)$\ell_3 = 2$

4　结束语

本文证明了星形弹性和粘弹性网络是指数稳定的,这为后续推广到一般网络的稳定性提供了可能性,未来可以通过改变弹性部分和粘弹性部分的分布比例,研究更加复杂网络的稳定性.

参 考 文 献

[1] RIVERA J E M, LAPA B C, BARRETO R. Decay rates for viscoelastic plates with memory [J]. Journal of Elasticity, 1996, 44(1): 61-87.

[2] MESSAOUDI S A. General decay of solutions of a viscoelastic equation [J]. Journal of Analysis and Applications, 2008, 341: 1457-1467.

[3] RIVERA J E M, SALVATIERRA A P. Asymptotic behavior of the energy in partially viscoelastic materials [J]. Quarterly of Applied Mathematics, 2001, 59(3): 557-578.

[4] RIVERA J E M, OQUENDO H P. Exponential stability to a contact problem of partially viscoelastic materials [J]. Journal of Elasticity and the Physical Science of Solids, 2001, 63(2): 87-111.

[5] DAGER R, ZUAZUA E. Controllability of star-shaped networks of strings [J]. Comptes Rendus de l'Académie des Sciences - Series I - Mathematics, 2001, 332(7): 621- 626.

[6] DAGER R. Observation and control of vibrations in tree-shaped networks of strings [J]. SIAM Journal on Control and Optimization, 2004, 43: 590-623.

[7] MERCIER D, REGNIER V. Control of a network of Euler-Bernoulli beams [J]. Journal of Mathematical Analysis and Applications, 2008, 342(2): 874-894.

[8] HAN Z J, XU G Q. Stabilization and Riesz basis of a star-shaped network of Timoshenko beams [J]. Journal of Dynamical and Control Systems, 2010, 16(2): 227-258.

[9] GUO Y N, XU G Q. Asymptotic stability and Riesz basis property for tree-shaped network of strings [J]. Journal of Systems Science and Complexity, 2011, 24(2): 225-252.

[10] XU G Q, YUNG S. Stability and riesz basis property of a star-shaped network of Euler-Bernoulli beams with joint damping [J]. Networks and Heterogeneous Media, 2017, 3(4): 723-747.

[11] AMMARI K, JELLOULI M. Stabilization of star-shaped networks of strings [J]. Differential and Integral Equations, 2004, 17: 11-12.

[12] DAFERMOS C M. Asymptotic stability in viscoelasticity [J]. Archive for Rational Mechanics and Analysis, 1970, 37: 297-308.

[13] PAZY A. Semigroup of linear operators and applications to partial differential equations [M]. New York: Springer, 1983.

[14] ZHANG Y X, XU G Q. Exponential and super stability of a wave network [J]. Acta Applicandae Mathematicae, 2013, 124(1): 19-41.

[15] MESSAOUDI S A. General decay of solutions of a viscoelastic equation [J]. Arabian Journal for Science and Engineering, 2008, 36(2): 1457-1467.

[16] 许跟起. Banach空间中线性算子理论 [M]. 北京: 学苑出版社, 2011.

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